FINITELY COLOURED ORDINALS

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Abstract. Two structures A and B are n-equivalent if player II has a winning strategy in the n-move Ehrenfeucht-Fraïssé game on A and B. Ordinals and m-coloured ordinals are studied up to n-equivalence for various values of m and n.

1. Introduction

Let A and B be coloured linear orders. We say that A is n-equivalent to B, written $A \equiv_n B$, if player II has a winning strategy in the n-move Ehrenfeucht-Fraïssé game on A and B. In [4] we established bounds on the least representatives of the n-equivalence classes of coloured linear orders in the special cases in which the ordering is finite, or the number of moves is at most 2. Here our focus is on the case of ordinals, with or without colours.

We briefly recall the material from [4] on coloured orderings and games that we need. A coloured linear ordering is a triple $(A, <, F)$ where $(A, <)$ is a linear order and F is a mapping from A onto a set C which we think of as a set of colours. We just write A instead of $(A, <, F)$ provided that the ordering and colouring are clear. In the n-move Ehrenfeucht-Fraïssé game on coloured linear orders A and B (or indeed any relational structures) players I and II play alternately, I moving first. On each move I picks an element of either structure (his choice does not have to be from the same structure on every move), and II responds by choosing an element of the other structure. After n moves, I and II between them have chosen elements $x_1, x_2, \ldots, x_n$ of A, and $y_1, y_2, \ldots, y_n$ of B, and player II *wins* if the map taking x_i to y_i for each i is an isomorphism (that is, it preserves the ordering and colour), and player I wins otherwise. Intuitively, I is trying to demonstrate that there is some difference between the structures, while player II is trying to show that they are at least reasonably similar. We say that A and B are *n-equivalent* and write $A \equiv_n B$, if II has a winning strategy. It is easy to see that $\equiv_n$ is an equivalence relation, and it is standard that for any n, there are only finitely many n-equivalence classes, so it is natural to enquire what their optimal representatives may be. The problem for general orderings seems to be quite hard, but with special conditions on the type of ordering or colouring, or the number of moves, some results can be obtained. If the ordering is an ordinal, then the notion of 'optimality' makes sense: a (coloured) ordinal is *optimal* if it is of least length in its n-equivalence class. This may still not be unique in the coloured case. If the ordering is finite, then we may take the lexicographically least; in the general case we would hope to make some canonical choice, for instance, exhibiting some eventual periodicity.

Already in [5], some information about the optimal representatives of n-equivalence classes of (monochromatic) ordinals is given (also see [1]). Rosenstein remarks (as

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an exercise) that every ordinal is $2n$-equivalent to some ordinal in the finite set

$$\{\omega^n \cdot a_n + \omega^{n-1} \cdot a_{n-1} + \ldots + \omega \cdot a_1 + a_0 : a_i < 2^{2n}, a_n \leq 1\}.$$

In section 2 we sharpen this result to give precise lists of all the optimal values for n-equivalence classes of ordinals, including the case where n is odd.

In section 3 we move on to consider the coloured case. By [4], we already understand the situation for 2 moves, and we now generalize this to more moves. Here we concentrate on giving some upper bounds for the optimal representatives, which certainly seem unnecessarily large, but at least all lie below the ordinal ω^ω.

Next we recall the notion of 'character' from [4], and the main result about characters. Assume that we have found representatives for the n-equivalence classes of certain m-coloured linearly ordered sets. We write the representative for A as $[A]_n$. In a coloured linear order A, the *n-character* of $a \in A$ having colour c is the ordered pair $\langle [A^{<a}]_n, [A^{>a}]_n \rangle$ (where $A^{<a} = \{x \in A : x < a\}$ and $A^{>a} = \{x \in A : x > a\}$). We let $\rho_n^c(A) = \{\langle [A^{<a}]_n, [A^{>a}]_n \rangle : a \in A \text{ is } c\text{-coloured}\}$, and if we wish to include the colour as part of the n-character of a, we may also write $\langle [A^{<a}]_n, [A^{>a}]_n \rangle_c$.

Theorem 1.1 ([4])**.** *$A \equiv_{n+1} B$ if and only if $\rho_n^c(A) = \rho_n^c(B)$ for all $c \in C$.*

If A and B are coloured linear orders, then $A + B$ stands for the concatenation of A and B, that is, we first assume (by replacing by copies if necessary) that A and B are disjoint, and we place all members of A to the left of all members of B. As a generalization of this, we may write $\sum\{A_i : i \in I\}$ for the concatenation of a family of (coloured) linear orders $\{A_i : i \in I\}$ indexed by a linear ordering I. When forming concatenations we would normally assume that all the orderings have the same colour set. We write $A \cdot B$ for the anti-lexicographic product, B 'copies of' A, to accord with the customary use for ordinals (and unlike [4], where *lexicographic* products are used). Note that here B is assumed monochromatic, and colours are assigned to members of $A \cdot B$ by means of the A-co-ordinate. The following result will be used without explicit reference.

Theorem 1.2. (i) *If $A \equiv_n B$, then $X+A+Y \equiv_n X+B+Y$ and $X\cdot A\cdot Y \equiv_n X\cdot B\cdot Y$.*
(ii) *If $A_i \equiv_n B_i$ for each $i \in I$, then $\sum\{A_i : i \in I\} \equiv_n \sum\{B_i : i \in I\}$.*

We conclude the introduction by quoting the following results which will be used throughout.

Lemma 1.3. *Let A and B be finite linear orderings. Then $A \equiv_n B$ if and only if ($|A| = |B| < 2^n - 1$) or ($|A|, |B| \geq 2^n - 1$).*

This is well known, but may be easily proved using characters.

Theorem 1.4 (Mostowski-Tarski)**.** *For any $n > 0$ and ordinal $\beta > 0$,*

(i) $\omega^n \equiv_{2n} \omega^n \cdot \beta$,
(ii) $\omega^n \not\equiv_{2n+1} \omega^n + \beta$,
(iii) $\omega^n \not\equiv_{2n+1} \omega^n \cdot \beta$, *for $\beta > 1$.*

See [5] for a proof. Note that (iii) shows that (i) is the best possible for player II (and (iii) is an immediate consequence of (ii)).

2. Optimal representatives for ordinals

We begin by remarking on the situation for $n = 1$ and 2, which was treated in [4]. For $n = 1$, all non-empty linear orders are n-equivalent. We therefore have two classes which can be represented by linear orders 0 and 1. For $n = 2$, it follows from Lemma 3.2 in [4] that a complete family of representatives is given by 0, 1, 2, 3 and ω (and any infinite ordinal is 2-equivalent to ω if it is a limit ordinal, and to 3 if it is a successor). The case $n = 0$ is degenerate, but still fits into the overall pattern; since there are no moves, all structures are equivalent, so there is one minimal representative, namely 0.

Many of our proofs will be by induction, which means that we shall concentrate on describing the first moves of the two players, and then appeal to the induction hypothesis. We usually write A and B for the two structures (or α and β), and x_1 and y_1 for the first elements chosen from A, B respectively. Subsequent moves played in A are $x_2, \ldots, x_n$ and in B are $y_2, \ldots, y_n$. Thus on each move, one of player I and player II plays x_i, and the other plays y_i, but which one plays which may vary during the game.

First we give the following two lemmas which throughout the paper will reduce the number of cases to be considered.

Lemma 2.1. *If $A = \omega^i \cdot \gamma_0$ and $B = \gamma_1 + \omega^j$ where $j < i \leq \frac{n}{2}$, then $A \not\equiv_n B$.*

Proof. Player I chooses $y_1 = \gamma_1 \in B$ so that $B^{>y_1} \cong \omega^j$ (or $B^{>y_1} = \emptyset$ if $j = 0$). Whatever $x_1 \in A$ player II plays, $A^{>x_1} \cong \omega^i \cdot \gamma_2$ for some $\gamma_2 > 0$. If $j = 0$, $A^{>x_1} \not\equiv_{n-1} B^{>y_1}$ is immediate. Otherwise, by Theorem 1.4(iii), $\omega^j \not\equiv_{2j+1} \omega^i \cdot \gamma_2$, and since $2j + 1 \leq n - 1$, I can therefore win in the remaining $n - 1$ moves. ■

Lemma 2.2. *Let $A = \omega^i \cdot a_i + \omega^{i-1} \cdot a_{i-1} + \ldots + \omega \cdot a_1 + a_0$ and $B = \omega^i \cdot b_i + \omega^{i-1} \cdot b_{i-1} + \ldots + \omega \cdot b_1 + b_0$. Then in any play of the n-move game on A and B in which player I starts by playing $x_1 = \omega^j \cdot \gamma_0$ for some ordinal $\gamma_0 > 0$ where $j < \frac{n}{2}$, unless player II plays $y_1 = \omega^j \cdot \gamma_1$ for some $\gamma_1 > 0$ then I can win the game in the remaining $n - 1$ moves.*

Proof. Supposing on the contrary that $B^{<y_1}$ has a final segment of order-type ω^r for some $r < j$ (possibly 0), we may write $A^{<x_1} \cong \omega^j \cdot \gamma_0$, $B^{<y_1} \cong \gamma_1 + \omega^r$, where $r < j \leq \frac{n-1}{2}$, and so by Lemma 2.1, $A^{<x_1} \not\equiv_{n-1} B^{<y_1}$, and player I wins. ■

Lemma 2.3. *Let n, m, i, k be integers such that $0 < i \leq \frac{n}{2}$.*

(i) *If $k \geq m = 2^{n-2i}$ then $\omega^i \cdot k \equiv_n \omega^i \cdot m$.*
(ii) *If $k > m$ and $m < 2^{n-2i}$ then $\omega^i \cdot k \not\equiv_n \omega^i \cdot m$.*

Proof. (i) We use induction on n. Since $0 < i \leq \frac{n}{2}$, $n \geq 2$. If $n = 2$, then $i = 1$ and $m = 1$. Now $\omega \cdot 1 \equiv_2 \omega \cdot k$ for any $k \geq 1$, giving the result.

So we assume the result for $n \geq 2$, and prove it for $n + 1$. Let $0 < i \leq \frac{n+1}{2}$, and $k \geq m = 2^{n+1-2i}$, with the object of showing that $\omega^i \cdot k \equiv_{n+1} \omega^i \cdot m$. If $i = \frac{n+1}{2}$, then n is odd, so by Theorem 1.4(i), $\omega^{\frac{n+1}{2}} \cdot k \equiv_{n+1} \omega^{\frac{n+1}{2}} \equiv_{n+1} \omega^{\frac{n+1}{2}} \cdot m$. So from now on we assume that $0 < i < \frac{n+1}{2}$.

Let $A = \omega^i \cdot k$ and $B = \omega^i \cdot m$. On his first move, player II may play so that if $A^{<x_1}$ has the form $\omega^i \cdot q_0 + \gamma$ where $\gamma < \omega^i$ and $0 \leq q_0 < 2^{n-2i}$, then $A^{<x_1} \cong B^{<y_1}$. (In other words, if x_1 is I's move, which satisfies this condition, then II can choose a corresponding y_1, and if y_1 is I's move, which satisfies this condition, then II can choose a corresponding x_1.) It follows that $A^{<x_1} \equiv_n B^{<y_1}$, and in this case, $A^{>x_1}$ and $B^{>y_1}$ have the forms $\omega^i \cdot q_1$ and $\omega^i \cdot q_2$ respectively, where $q_1, q_2 \geq 2^{n-2i}$. By induction hypothesis, $A^{>x_1} \equiv_n B^{>y_1}$, so II can win the $(n+1)$-move game by calling on his strategies on the left and right of x_1, y_1 as required in the remaining n moves.

In a similar way, player II may play on his first move so that if $A^{>x_1}$ has the form $\omega^i \cdot r_0$ where $1 \leq r_0 \leq 2^{n-2i}$, then $A^{>x_1} \cong B^{>y_1}$. Here $A^{>x_1} \equiv_n B^{>y_1}$, and in this case, II may also ensure that $A^{<x_1}$ and $B^{<y_1}$ have the form $\omega^i \cdot r_1 + \gamma$ and $\omega^i \cdot r_2 + \gamma$ respectively, where $r_1, r_2 \geq 2^{n-2i}$ and therefore by the induction hypothesis, it follows that $A^{<x_1} \equiv_n B^{<y_1}$. Once more this provides II with a winning strategy in the $(n+1)$-move game.

Finally, player II may play on his first move so that if $A^{<x_1}$ has the form $\omega^i \cdot s_0 + \gamma$ where $2^{n-2i} \leq s_0 < k - 2^{n-2i}$ and $\gamma < \omega^i$, then $B^{<y_1} \cong \omega^i \cdot 2^{n-2i} + \gamma$. Again using the induction hypothesis, $A^{<x_1} \equiv_n B^{<y_1}$. In this case $A^{>x_1}$ and $B^{>y_1}$ have the form $\omega^i \cdot s_1$ where $s_1 \geq 2^{n-2i}$, and $\omega^i \cdot 2^{n-2i}$. By the induction hypothesis, we deduce that $A^{>x_1} \equiv_n B^{>y_1}$, and again II wins.

(ii) Again using induction, for the basis case, $n = 2$, in which case $i = 1$ and $m = 0$ so the result is immediate.

Now assume the result for n, and let $0 < i \leq \frac{n+1}{2}$, $k > m$, and $m < 2^{n+1-2i}$. Since the result is immediate for $m = 0$, we assume that $m \neq 0$ which means that $i < \frac{n+1}{2}$, so $i \leq \frac{n}{2}$. Let r be the integer part $[\frac{k}{2}]$ of $\frac{k}{2}$, and on his first move, I plays $x_1 = \omega^i \cdot r$. Then $A^{<x_1} \cong \omega^i \cdot r$ and $A^{>x_1} \cong \omega^i(k-r)$. Suppose that II plays y_1. By Lemma 2.2 we may suppose that $y_1 = \omega^i \cdot s$ for some s. Player I can now win in the remaining n moves on the left or right, provided that $A^{<x_1} \not\equiv_n B^{<y_1}$ or $A^{>x_1} \not\equiv_n B^{>y_1}$. As remarked above, $0 < i \leq \frac{n}{2}$.

If $s < r$ and $s < 2^{n-2i}$ then by induction hypothesis, $A^{<x_1} \not\equiv_n B^{<y_1}$. If however, $s < r$ and $s \geq 2^{n-2i}$, then $r > 2^{n-2i}$ which implies that $k > 2^{n+1-2i}$, and hence that $k-r \geq 2^{n-2i} > m-s$. Therefore $A^{>x_1} \not\equiv_n B^{>y_1}$, again by the induction hypothesis. Otherwise $s \geq r$, which implies that $m-s < k-r$. Now $m-s \geq 2^{n-2i}$ is impossible, since it implies that $k-r > 2^{n-2i}$ so also $r \geq 2^{n-2i}$, giving $s, m-s \geq 2^{n-2i}$ and $m \geq 2^{n+1-2i}$, contrary to supposition. The conclusion is that $m-s < 2^{n-2i}$, which again gives $A^{>x_1} \not\equiv_n B^{>y_1}$. ■

We write t for the integer part $[\frac{n}{2}]$ of $\frac{n}{2}$.

Corollary 2.4. *If $n > 0$ then every ordinal is n-equivalent to some ordinal in the finite set* $\Omega_n = \{\omega^t \cdot a_t + \omega^{t-1} \cdot a_{t-1} + \ldots + \omega \cdot a_1 + a_0 : a_i \leq 2^{n-2i}\}$.

Proof. First suppose that n is even. Using Cantor normal form we may write any ordinal α in the form $\alpha = \omega^t \cdot \alpha^* + \omega^{t-1} \cdot b_{t-1} + \ldots + \omega \cdot b_1 + b_0$ where α^* is an ordinal, and $b_i \in \omega$. By Theorem 1.4(i), $\omega^t \cdot \alpha^* \equiv_n \omega^t$ if $\alpha^* \neq 0$, and by Lemma 2.3(i), $\omega^i \cdot b_i \equiv_n \omega^i \cdot a_i$ where $a_i = \min(b_i, 2^{n-2i})$. Finally letting $a_t = \min(\alpha^*, 1)$, we find that $\alpha \equiv_n \omega^t \cdot a_t + \omega^{t-1} \cdot a_{t-1} + \ldots + \omega \cdot a_1 + a_0 \in \Omega_n$.

The proof for odd n is similar except that we let $a_t = \min(\alpha^*, 2)$. Note that we cannot appeal to Theorem 1.4(i) directly this time to show that $\omega^t \cdot \alpha^* \equiv_n \omega^t \cdot 2$ for $\alpha^* \geq 2$, and instead follow a direct proof. Player II may play so that $x_1 = \omega^t \cdot q_1 + \gamma$, $y_1 = \omega^t \cdot q_2 + \gamma$, where $q_1 < \alpha^*$, $q_2 = 0$ or 1 and $q_1 = 0 \Leftrightarrow q_2 = 0$. The facts that $A^{<x_1} \equiv_{n-1} B^{<y_1}$ and $A^{>x_1} \equiv_{n-1} B^{>y_1}$ follow from Theorem 1.4(i). ■

Corollary 2.5. (i) *If n is even, then any ordinal is n-equivalent to some ordinal* $\leq \omega^{\frac{n}{2}} \cdot 2$,

(ii) *If n is odd, then any ordinal is n-equivalent to some ordinal* $\leq \omega^{\frac{n-1}{2}} \cdot 3$.

We now give a list, without proof, of the minimal n-equivalence class representatives for $n = 3$ and 4. Proofs that these are the correct lists are given in [3], and they form the basis for the general result we prove in Theorem 2.13, which yields these two lists as special cases.

The minimal 3-equivalence class representatives for all ordinals are

$0, 1, 2, 3, 4, 5, 6, 7,$
$\omega, \omega+1, \omega+2, \omega+3, \omega+4,$
$\omega \cdot 2, \omega \cdot 2+1, \omega \cdot 2+2, \omega \cdot 2+3.$

The minimal 4-equivalence class representatives for all ordinals are

$0, 1, 2, \ldots, 15,$
$\omega, \omega+1, \omega+2, \omega+3, \omega+4, \ldots, \omega+12,$
$\omega \cdot 2, \omega \cdot 2+1, \omega \cdot 2+2, \omega \cdot 2+3, \ldots, \omega \cdot 2+12,$

$\omega\cdot 3, \omega\cdot 3+1, \omega\cdot 3+2, \ldots, \omega\cdot 3+12,$
$\omega\cdot 4, \omega\cdot 4+1, \omega\cdot 4+2, \omega\cdot 4+3,$
$\omega^2, \omega^2+1, \omega^2+2, \omega^2+3.$

Rosenstein's list of $2n$-equivalence class representatives that we quoted in the introduction includes some redundancies, and indeed we have already illustrated this in Corollary 2.4. We shall show that even this list can be improved, and give explicit lists of representatives of ordinals up to n-equivalence by making use of the patterns seen in generating the two lists just given. Thus if Ω_n is the set of n-equivalence class representatives provided by Corollary 2.4, we shall find $\Omega'_n \subseteq \Omega_n$ that contains no redundant elements.

The following result generalizes an exercise in [5] page 106.

Lemma 2.6. *For all even $n \geq 4$ and ordinals $\alpha \geq 3$,*

(i) $\omega^{\frac{n}{2}-1}(\alpha+1) \equiv_n \omega^{\frac{n}{2}-1}\cdot 4$,
(ii) $\omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1} \not\equiv_n \omega^{\frac{n}{2}-1}\cdot 3$.

Proof. (i) We describe a winning strategy for player II. Let us write $A = \omega^{\frac{n}{2}-1}(\alpha+1)$ and $B = \omega^{\frac{n}{2}-1}\cdot 4$. Player II may move on his first move so that if x_1 is in the 0, 1 or last copy of $\omega^{\frac{n}{2}-1}$ in A, then y_1 is in the corresponding copy of B, and if x_1 is in any other copy of $\omega^{\frac{n}{2}-1}$ in A, then y_1 is in the third copy (numbered by 2) of $\omega^{\frac{n}{2}-1}$ in B. Furthermore, player II may play so that x_1 and y_1 are the corresponding points in those copies.

The outcomes in these four cases are as follows:

$A^{<x_1} \cong B^{<y_1}$ and $A^{>x_1} \cong \omega^{\frac{n}{2}-1}(\alpha+1)$, $B^{>y_1} \cong \omega^{\frac{n}{2}-1}\cdot 4$,
$A^{<x_1} \cong B^{<y_1}$ and $A^{>x_1} \cong \omega^{\frac{n}{2}-1}(\alpha_1+1)$, $B^{>y_1} \cong \omega^{\frac{n}{2}-1}\cdot 3$, where $\alpha = 1+\alpha_1$,
$A^{<x_1} \cong \omega^{\frac{n}{2}-1}\cdot\alpha+\gamma$, $B^{<y_1} \cong \omega^{\frac{n}{2}-1}\cdot 3+\gamma$ and $A^{>x_1} \cong B^{>y_1}$,
$A^{<x_1} \cong \omega^{\frac{n}{2}-1}\cdot\alpha_1+\gamma$, $B^{<y_1} \cong \omega^{\frac{n}{2}-1}\cdot 2+\gamma$, for some $\gamma < \omega^{\frac{n}{2}-1}$, $A^{>x_1} \cong \omega^{\frac{n}{2}-1}(\alpha_2+1)$, $B^{>y_1} \cong \omega^{\frac{n}{2}-1}\cdot 2$, where $\alpha_1+\alpha_2 = \alpha$, $2 \leq \alpha_1 < \alpha$.

In each case Player II has a winning strategy in the remaining $n-1$ moves, whether player I plays on the left or right of the first moves. When the relevant structures are isomorphic this is immediate. Otherwise, player II may play so that x_2 and y_2 are corresponding points of some copies of $\omega^{\frac{n}{2}-1}$ (or of a 'γ' part), and if one of the copies is the first one, then so is the other; for the remaining $n-2$ moves, player II uses Theorem 1.4(i) to win.

(ii) Let $A = \omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1}$ and $B = \omega^{\frac{n}{2}-1}\cdot 3$. On his first move, player I plays the first point x_1 of copy number 3 of $\omega^{\frac{n}{2}-1}$ in A, and II responds by playing the first point y_1 of the ith copy of $\omega^{\frac{n}{2}-1}$ in B, $0 < i < 3$ (if he chooses 0 or a non-first point, then he loses by Lemma 2.2). If $i = 1$ then from now on I plays on the left of x_1, y_1, or if $i = 2$ he plays on the right of x_1, y_1, in each case winning using Theorem 1.4(iii). ■

Lemma 2.7. *Let $m \geq 4$, $0 < i \leq \frac{n-1}{2}$, and k be an ordinal.*

(i) *If $k \geq 2^{n-2i}$, then $\omega^i\cdot k + \omega^{i-1}\cdot m \equiv_n \omega^i(2^{n-2i}-1) + \omega^{i-1}\cdot m$,*
(ii) *If $l < 2^{n-2i}$ and $l < k$, then $\omega^i\cdot k + \omega^{i-1}\cdot 3 \not\equiv_n \omega^i\cdot l + \omega^{i-1}\cdot 3$.*

Proof. (i) We use induction. Notice that as $\frac{n-1}{2} \geq 1$, we have $n \geq 3$. If $n = 3$, then $i = 1$, so we have to check that $\omega\cdot k+m \equiv_3 \omega+m$ for $k \geq 2$. We find that $\omega\cdot k+m$ and $\omega+m$ both have character set $\{\langle 0,3\rangle, \langle 1,3\rangle, \langle 2,3\rangle, \langle 3,3\rangle, \langle \omega,3\rangle, \langle 3,2\rangle, \langle 3,1\rangle, \langle 3,0\rangle\}$, and so they are 3-equivalent.

Now assume the result holds for $n \geq 3$ and we prove it for $n+1$. So we consider $A = \omega^i\cdot k + \omega^{i-1}\cdot m$ and $B = \omega^i(2^{n+1-2i}-1) + \omega^{i-1}\cdot m$ where $0 < i \leq \frac{n}{2}$, and $k \geq 2^{n+1-2i}$, and we have to show that player II has a winning strategy in the $(n+1)$-move game. First note that II can play in such a way that x_1 lies in the final $\omega^{i-1}\cdot m$ segment of A if and only if y_1 lies in the final $\omega^{i-1}\cdot m$ segment of B

and then they are corresponding points. By Lemma 2.3(i), $\omega^i \cdot k \equiv_n \omega^i \cdot 2^{n-2i} \equiv_n \omega^i(2^{n+1-2i}-1)$, and this provides a winning strategy for player II in the remaining n moves, since this shows that $A^{<x_1} \equiv_n B^{<y_1}$ (and $A^{>x_1} \equiv_n B^{>y_1}$ because they are isomorphic).

Now supposing that x_1 and y_1 do not lie in the final part, the first case is where $i = \frac{n}{2}$. Then n is even, and $A = \omega^{\frac{n}{2}} \cdot k + \omega^{\frac{n}{2}-1} \cdot m$, $B = \omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1} \cdot m$. Player II can play so that one of the following holds:

$x_1 = y_1 < \omega^{\frac{n}{2}}$, in which case $A^{<x_1} \cong B^{<y_1}$ so $A^{<x_1} \equiv_n B^{<y_1}$, and $A^{>x_1} \cong \omega^{\frac{n}{2}} \cdot k + \omega^{\frac{n}{2}-1} \cdot m$, $B^{>y_1} \cong \omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1} \cdot m$, which are n-equivalent by Theorem 1.4(i).

$x_1 = \omega^{\frac{n}{2}} \cdot q_1 + \gamma$ and $y_1 = \omega^{\frac{n}{2}} + \gamma$ where $1 \le q_1 < k$ and $\gamma < \omega^{\frac{n}{2}-1}$. Then $A^{<x_1} \cong \omega^{\frac{n}{2}} \cdot q_1 + \gamma$ and $B^{<y_1} \cong \omega^{\frac{n}{2}} + \gamma$ so $A^{<x_1} \equiv_n B^{<y_1}$ by Theorem 1.4(i) and $A^{>x_1} \cong \omega^{\frac{n}{2}}(k-q_1)+\omega^{\frac{n}{2}-1}\cdot m$ (where if k is infinite, $k-q_1 = k$) and $B^{>y_1} \cong \omega^{\frac{n}{2}-1}\cdot m$ so $A^{>x_1} \equiv_n B^{>y_1}$ by Lemmas 2.6(i) and 2.3(i).

$x_1 = \omega^{\frac{n}{2}} \cdot q_1 + \omega^{\frac{n}{2}-1} \cdot q_2 + \gamma$ and $y_1 = \omega^{\frac{n}{2}-1} \cdot 4 + \gamma$ where $1 \le q_1 < k$, $1 \le q_2 < \omega$, and $\gamma < \omega^{\frac{n}{2}-1}$. Then $A^{<x_1} \cong \omega^{\frac{n}{2}} \cdot q_1 + \omega^{\frac{n}{2}-1} \cdot q_2 + \gamma$ and $B^{<y_1} \cong \omega^{\frac{n}{2}-1} \cdot 4 + \gamma$ so $A^{<x_1} \equiv_n B^{<y_1}$ by Lemmas 2.6(i) and 2.3(i), and $A^{>x_1} \cong \omega^{\frac{n}{2}}(k-q_1) + \omega^{\frac{n}{2}-1} \cdot m$ and $B^{>y_1} \cong \omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1} \cdot m$ so $A^{>x_1} \equiv_n B^{>y_1}$ by Theorem 1.4(i).

Otherwise, $0 < i < \frac{n}{2}$ and hence $0 < i \le \frac{n-1}{2}$, so by induction hypothesis, $\omega^i \cdot 2^{n-2i} + \omega^{i-1} \cdot m \equiv_n \omega^i(2^{n-2i}-1) + \omega^{i-1} \cdot m$. Player II can play so that if $x_1 < \omega^i(2^{n-2i}+1)$ then $x_1 = y_1$, and if $\omega^i(2^{n-2i}+1) \le x_1 < \omega^i \cdot k$ then for some γ, and finite $r_1, r_2 \ge 2^{n-2i}$, $A^{<x_1} \cong \omega^i \cdot r_1 + \gamma$, $B^{<y_1} \cong \omega^i \cdot r_2 + \gamma$, and $A^{>x_1} \cong B^{>y_1}$.

In the first case, $A^{>x_1} \cong \omega^i \cdot r_3 + \omega^{i-1} \cdot m$ and $B^{>y_1} \cong \omega^i \cdot r_4 + \omega^{i-1} \cdot m$ where $r_3 \ge 2^{n-2i}$ and $r_4 \ge 2^{n-2i}-1$, so $A^{<x_1} = B^{<y_1}$ giving $A^{<x_1} \equiv_n B^{<y_1}$ and $A^{>x_1} \equiv_n B^{>y_1}$ by induction hypothesis, which provides a winning strategy for II in the $(n+1)$-move game. In the second case, $A^{<x_1} \equiv_n B^{<y_1}$ by Lemma 2.3(i) and $A^{>x_1} \cong B^{>y_1}$, so II again wins.

In all cases we deduce that $A \equiv_{n+1} B$.

(ii) We use induction. As above, $n \ge 3$. If $n = 3$ then $i = 1$, and so we have to show that $\omega \cdot l + 3 \not\equiv_3 \omega \cdot k + 3$ for $l \le 1$ and $k > l$. This is verified by consideration of 2-characters. If $l = 1$ then $k \ge 2$, so $\omega \cdot k + 3$ has $\langle \omega, 3 \rangle$ as a 2-character, but $\omega \cdot l + 3$ does not. If $l = 0$ then $\omega \cdot k + 3$ has $\langle \omega, 2 \rangle$ as a 2-character, but $\omega \cdot l + 3$ does not.

For the induction step we assume the result for $n \ge 3$ and show that I has a winning strategy in the $(n+1)$-move game on $A = \omega^i \cdot k + \omega^{i-1} \cdot 3$ and $B = \omega^i \cdot l + \omega^{i-1} \cdot 3$ where $0 < i \le \frac{n}{2}$, $l < 2^{n+1-2i}$, and $l < k$. In the first case, $i = \frac{n}{2}$, so that n is even, and we have to show that $A = \omega^{\frac{n}{2}} \cdot k + \omega^{\frac{n}{2}-1} \cdot 3 \not\equiv_{n+1} B = \omega^{\frac{n}{2}} \cdot l + \omega^{\frac{n}{2}-1} \cdot 3$ where $l = 1$ or 0, and $k > l$. Let I play $x_1 = \omega^{\frac{n}{2}} \in A$ on his first move. By Lemma 2.2, noting that $\frac{n}{2} < \frac{n+1}{2}$, we may suppose that II's reply y_1 is a non-zero multiple of $\omega^{\frac{n}{2}}$. Since $l \le 1$, this implies that $l = 1$ (and so $k \ge 2$) and $y_1 = \omega^{\frac{n}{2}} \in B$, and I now plays $x_2 = \omega^{\frac{n}{2}} \cdot 2 \in A$. By Lemma 2.2 again, II plays $y_2 = \omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1}$ or $\omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1} \cdot 2$ in B. If $y_2 = \omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1}$, player I wins on the intervals (x_1, x_2) and (y_1, y_2) using $\omega^{\frac{n}{2}-1} \not\equiv_{n-1} \omega^{\frac{n}{2}}$ and if $y_2 = \omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1} \cdot 2$, he wins to the right of x_2 and y_2 using $\omega^{\frac{n}{2}}(k-2) + \omega^{\frac{n}{2}-1} \cdot 3 \not\equiv_{n-1} \omega^{\frac{n}{2}-1}$, in each case appealing to Theorem 1.4(iii).

Now we suppose that $i < \frac{n}{2}$, and let $q_1 = \min(2^{n-2i}, [\frac{k}{2}])$. Player I plays $x_1 = \omega^i \cdot q_1 \in A$, and by again appealing to Lemma 2.2, we may assume that II's response is of the form $y_1 = \omega^i \cdot q_2$ for some q_2 with $1 \le q_2 \le l$. Then $A^{<x_1} = \omega^i \cdot q_1$, $B^{<y_1} = \omega^i \cdot q_2$, $A^{>x_1} \cong \omega^i(k-q_1) + \omega^{i-1} \cdot 3$, and $B^{>y_1} \cong \omega^i(l-q_2) + \omega^{i-1} \cdot 3$. If $q_2 < q_1$ then by Lemma 2.3(ii), $\omega^i \cdot q_1 \not\equiv_n \omega^i \cdot q_2$, so player I can win by playing on the left of x_1 and y_1, and if $q_2 \ge q_1$, then $k - q_1 > l - q_2$ and he can play on the right of x_1 and y_1 using $\omega^i(k-q_1) + \omega^{i-1} \cdot 3 \not\equiv_n \omega^i(l-q_2) + \omega^{i-1} \cdot 3$, which follows by the induction hypothesis, since $l - q_2 < 2^{n-2i}$. For if $k \geq 2^{n+1-2i}$, then $q_2 \geq q_1 = 2^{n-2i}$, so $l - q_2 < 2^{n+1-2i} - 2^{n-2i} = 2^{n-2i}$ and if $k < 2^{n+1-2i}$, then $q_1 = [\frac{k}{2}]$, so $l - q_2 < k - q_1 \leq \frac{k+1}{2} \leq 2^{n-2i}$. ∎

Corollary 2.8. *If $n \geq 4$ is even and $t = \frac{n}{2}$, then*

(i) $\omega^t + \omega^{t-2} \cdot m \equiv_n \omega^{t-1} \cdot 3 + \omega^{t-2} \cdot m$ *for* $m \geq 4$,
(ii) $\omega^t + \omega^{t-2} \cdot l \not\equiv_n \omega^{t-1} \cdot 3 + \omega^{t-2} \cdot l$ *for* $l < 4$.

This follows from Lemma 2.7 on taking $k = \omega$ and $i = t - 1$.

Lemma 2.9. *Let α and β be n-equivalent ordinals such that $\alpha = \omega^j \cdot \gamma$ for some $\gamma \geq 1$. If $i \leq \frac{n-3}{2}$, $i < j \leq \frac{n}{2}$ and $k \geq 2^{n-2i} - 4 > m$ is finite, then*

(i) $\alpha + \omega^i \cdot k \equiv_n \beta + \omega^i(2^{n-2i} - 4)$,
(ii) $\alpha + \omega^i \cdot k \not\equiv_n \beta + \omega^i \cdot m$.

Proof. (i) Observe that as $i \leq \frac{n-3}{2}$, $n - 2i \geq 3$, so $2^{n-2i} - 4 \geq 4$.

We use induction. When $n = 3$, i must be 0, and we have to show that $\alpha + k \equiv_3 \beta + 4$ for $k \geq 4$, which holds since these two ordinals have the same 2-characters (as $\alpha > 0$ is a limit ordinal, and hence so is β since $\alpha \equiv_3 \beta$).

For the induction step, assume the result for $n \geq 3$, and let $A = \alpha + \omega^i \cdot k$, $B = \beta + \omega^i(2^{n+1-2i} - 4)$, where $\alpha \equiv_{n+1} \beta$, $i \leq \frac{(n+1)-3}{2} = \frac{n}{2} - 1$, and $k \geq 2^{n+1-2i} - 4$, and we show that $A \equiv_{n+1} B$. If $i = \frac{n}{2} - 1$ then n is even and $A = \alpha + \omega^i \cdot k$, $B = \beta + \omega^i \cdot 4$, where $k \geq 4$. Then player II can play so that for some $\gamma < \omega^i$,

$x_1 \in \alpha \Leftrightarrow y_1 \in \beta$ and II has used his winning strategy in the $(n+1)$-move game on α and β,

or $x_1 = \alpha + \omega^i \cdot r + \gamma$, $y_1 = \beta + \omega^i \cdot s + \gamma$, where $\gamma < \omega^i$ and $r = s = 0$ or $k - r = 4 - s < 4$,

or $x_1 = \alpha + \omega^i \cdot r + \gamma$ and $y_1 = \omega^i \cdot 4 + \gamma$ where $0 < r \leq k - 4$.

In these four cases we find that for some $\alpha_1 \equiv_n \beta_1$,

$A^{>x_1} \cong \alpha_1 + \omega^i \cdot k$, $B^{>y_1} \cong \beta_1 + \omega^i \cdot 4$, and $A^{>x_1} \equiv_n B^{>y_1}$ by Lemma 2.3(i),

if $r = s = 0$ then $A^{>x_1} \cong \omega^i \cdot k$, $B^{>y_1} \cong \omega^i \cdot 4$, so again $A^{>x_1} \equiv_n B^{>y_1}$ by Lemma 2.3(i),

if $k - r = 4 - s < 4$, then $A^{>x_1} \cong B^{>y_1}$, and as $r, s > 0$, $A^{<x_1} \equiv_n B^{<y_1}$ by Lemmas 2.6(i) and 2.3(i),

if $0 < r \leq k-4$, then $A^{<x_1} \cong \alpha + \omega^i \cdot r + \gamma$, $B^{<y_1} \cong \omega^i \cdot 4 + \gamma$, and $A^{>x_1} \cong \omega^i(k-r)$, $B^{>y_1} \cong \beta + \omega^i \cdot 4$, so $A^{<x_1} \equiv_n B^{<y_1}$ and $A^{>x_1} \equiv_n B^{>y_1}$ both follow from Lemmas 2.6(i) and 2.3(i).

If $i < \frac{n}{2} - 1$ then $i \leq \frac{n-3}{2}$. We observe that we may write

$A = \alpha + \omega^i(2^{n-2i} - 4) + \omega^i(k - (2^{n+1-2i} - 4)) + \omega^i \cdot 2^{n-2i}$ and
$B = \beta + \omega^i(2^{n-2i} - 4) + \qquad\qquad\qquad\qquad\qquad\qquad \omega^i \cdot 2^{n-2i}$.

Player II can play so that

$x_1 \in \alpha \Leftrightarrow y_1 \in \beta$ and he has used his winning strategy in the $(n+1)$-move game on α and β,

or $x_1 = \alpha + \gamma$ where $\gamma < \omega^i(2^{n-2i} - 3)$, and $y_1 = \beta + \gamma$,

or $x_1 = \alpha + \omega^i \cdot q_1 + \gamma$ where $\gamma < \omega^i$, $2^{n-2i} - 4 < q_1 \leq k - 2^{n-2i}$, and $y_1 = \beta + \omega^i(2^{n-2i} - 4) + \gamma$,

or $x_1 = \alpha + \omega^i \cdot q_2 + \gamma$ and $y_1 = \beta + \omega^i \cdot q_3 + \gamma$ where $\gamma < \omega^i$, $q_2 > k - 2^{n-2i}$, $q_3 > 2^{n-2i} - 4$ and $A^{>x_1} \cong B^{>y_1} \leq \omega^i \cdot 2^{n-2i}$.

The first case is as above.

In the second case, $A^{<x_1} \equiv_n B^{<y_1}$. Also, we see that $A^{>x_1} \cong \omega^i \cdot r_1$ and $B^{>y_1} \cong \omega^i \cdot r_2$ where $r_1 \geq k - (2^{n-2i} - 4)$ and $r_2 \geq 2^{n-2i}$. We note that $k - (2^{n-2i} - 4) \geq 2^{n-2i}$ since $k \geq 2^{n+1-2i} - 4$. Therefore $A^{>x_1} \equiv_n B^{>y_1}$ by Lemma 2.3(i).

In the third case, by the induction hypothesis, $A^{<x_1} \equiv_n B^{<y_1}$. In addition, $A^{>x_1} \cong \omega^i \cdot s$ where $s \geq 2^{n-2i}$ and $B^{>y_1} \cong \omega^i \cdot 2^{n-2i}$. By Lemma 2.3(i), $A^{>x_1} \equiv_n B^{>y_1}$.

In the final case, $A^{>x_1} \equiv_n B^{>y_1}$, and $A^{<x_1} \equiv_n B^{<y_1}$ by the induction hypothesis.

(ii) We use induction, with $n = 3$ as the the basis case. Here, $i = 0$ and $m < 4 \leq k$, and we have to show that $\alpha + k \not\equiv_3 \beta + m$. Since α and β are limit ordinals, $\alpha + k$ exhibits the 2-character $\langle \omega, 3 \rangle$, but $\beta + m$ does not, so they are 3-inequivalent.

Now assume the result for $n \geq 3$ and we prove it for $n+1$. Let $A = \alpha + \omega^i \cdot k$ and $B = \beta + \omega^i \cdot m$ where $m < 2^{n+1-2i} - 4 \leq k$, $\alpha \equiv_{n+1} \beta$, and $i \leq \frac{(n+1)-3}{2} = \frac{n}{2} - 1$. Then A can be written in the form $\alpha + \omega^i(2^{n-2i} - 4) + \omega^i \cdot q$ where $q \geq 2^{n-2i}$.

Case 1: $i = 0$.

Player I plays $x_1 = \alpha + 2^n - 4$. Then $|A^{>x_1}| \geq 2^n - 1$ so if II plays y_1 and $|B^{>y_1}| < 2^n - 1$ then I wins (playing on the right) by the finite case. If $|B^{>y_1}| \geq 2^n - 1$ and $y_1 \geq \beta$, then $A^{<x_1} \cong \alpha + 2^n - 4$ and $B^{<y_1} \cong \beta + r$ where $r \leq m - 2^n < 2^n - 4$, so I wins in the remaining n moves playing on the left using the induction hypothesis.

Case 2: $0 < i < \frac{n}{2} - 1$.

Player I plays $x_1 = \alpha + \omega^i(2^{n-2i} - 4)$. By Lemma 2.2 we may assume that II plays a non-trivial multiple y_1 of ω^i.

If $y_1 \geq \beta$ and $B^{>y_1} < \omega^i \cdot 2^{n-2i}$, then $A^{>x_1} \not\equiv_n B^{>y_1}$ by Lemma 2.3(ii), and I wins.

If $y_1 \geq \beta$, and $B^{>y_1} \geq \omega^i \cdot 2^{n-2i}$, then $B^{<y_1} \cong \beta + \omega^i \cdot s_1$ where $s_1 < 2^{n-2i} - 4$ and $A^{<x_1} \not\equiv_n B^{<y_1}$ by induction hypothesis.

If however $y_1 < \beta$, then player I plays $y_2 = \beta$ on his second move. By Lemma 2.2 we may suppose that II plays a multiple $x_2 = x_1 + \omega^i \cdot s_2$ of ω^i, with $s_2 > 0$. Now I plays $x_3 = x_1 + \omega^i(s_2 - 1)$, and whatever y_3 II plays, $(x_3, x_2) \cong \omega^i$ and $(y_3, y_2) \cong \omega^j \cdot s_3$ for some $s_3 > 0$. By Theorem 1.4(iii), $\omega^i \not\equiv_{2i+1} \omega^j \cdot s_3$, and since $2i + 1 \leq n - 2$, I wins.

Case 3: $i = \frac{n}{2} - 1$. Then $2^{n+1-2i} - 4 = 4$ so $m < 4 \leq k$. Player I chooses $x_1 = \alpha + \omega^i(k - 3) \in A$, so that $A^{>x_1} \cong \omega^i \cdot 3$. By Lemma 2.2 we may assume that II picks a multiple y_1 of ω^i. If $y_1 > \beta$ then $A^{>x_1} \not\equiv_n B^{>y_1}$ follows from Lemma 2.3(ii). If $y_1 = \beta$, I plays $x_2 < x_1$ so that $(x_2, x_1) \cong \omega^i$, and (y_2, y_1) must be a multiple of ω^j, so $(x_2, x_1) \not\equiv_{2i+1} (y_2, y_1)$ by Theorem 1.4(iii), and as $2i + 1 = n - 1$, I wins. Finally, if $y_1 < \beta$, $A^{>x_1} \cong \omega^{\frac{n}{2}-1} \cdot 3$ and $B^{>y_1} \cong \omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1}$, and so $A^{>x_1} \not\equiv_n B^{>y_1}$ by Lemma 2.6(ii). ■

We next give an inductive generalization of Lemma 2.7.

Lemma 2.10. *If* $0 < r \leq i < \frac{n}{2}$, *then*

(i) $\omega^i \cdot 2^{n-2i} + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot 4 \equiv_n$
$\omega^i(2^{n-2i} - 1) + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot 4$,

(ii) *if* $b_j \leq 3$ *for all* $j < i$, *or for some* $k < i$, $b_j \leq 3$ *for all* j *such that* $k \leq j < i$ *and* $b_k \leq 2$, *then* $\omega^i \cdot 2^{n-2i} + \omega^{i-1} \cdot a_{i-1} + \ldots + \omega \cdot a_1 + a_0 \not\equiv_n$
$\omega^i(2^{n-2i} - 1) + \omega^{i-1} \cdot b_{i-1} + \ldots + \omega \cdot b_1 + b_0$.

Proof. (i) We use induction on n. The case $r = 1$ is covered by Lemma 2.7(i), so we suppose that $r \geq 2$, which means that there is at least one term in the sum having a coefficient of 3. Let $A = \omega^i \cdot 2^{n-2i} + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot 4$ and $B = \omega^i(2^{n-2i} - 1) + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot 4$.

We first consider the case where $i = \frac{n-1}{2}$, in which case n is odd, and $A = \omega^i \cdot 2 + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot 4$ and $B = \omega^i + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot 4$. Here we use induction on r. Player II can play so that one of the following holds:

$x_1 = y_1 < \omega^i$ and so $A^{<x_1} \equiv_{n-1} B^{<y_1}$, and $A^{>x_1} \cong \omega^i \cdot 2 + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot 4$ and $B^{>y_1} \cong \omega^i + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot 4$, and so $A^{>x_1} \equiv_{n-1} B^{>y_1}$ by Theorem 1.4(i).

$\omega^i \leq x_1 = y_1 < \omega^i + \omega^{i-1}$, giving $A^{<x_1} \equiv_{n-1} B^{<y_1}$, and in addition $A^{>x_1} \cong \omega^i + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot 4$ and $B^{>y_1} \cong \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot 4$.

By Lemma 2.6(i), $\omega^i + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot 4 \equiv_{n-1} \omega^{i-1} \cdot 4 + \omega^{i-2} \cdot 3 + \ldots + \omega^{i-r} \cdot 4$, so we just have to check that $\omega^{i-1} \cdot 4 + \omega^{i-2} \cdot 3 + \ldots + \omega^{i-r} \cdot 4 \equiv_{n-1} \omega^{i-1} \cdot 3 + \omega^{i-2} \cdot 3 + \ldots + \omega^{i-r} \cdot 4$. Player II may play so that $x_2 = y_2 < \omega^{i-1} \cdot 3$, or $x_2 = \omega^{i-1} + y_2 \geq \omega^{i-1} \cdot 3$. To make sure that this works, we have to check in the first case that $A^{>x_2} \equiv_{n-2} B^{>y_2}$, and in the second case, that $(x_1, x_2) \equiv_{n-2} (y_1, y_2)$. The former requires that $\omega^{i-1} \cdot 2 + \omega^{i-2} \cdot 3 + \ldots + \omega^{i-r} \cdot 4 \equiv_{n-2} \omega^{i-1} + \omega^{i-2} \cdot 3 + \ldots + \omega^{i-r} \cdot 4$, which follows from the induction hypothesis (on r), as $i - r = (i-1) - (r-1)$, and the second says that $\omega^{i-1} \cdot 3 \equiv_{n-2} \omega^{i-1} \cdot 2$, which follows from Lemma 2.3(i).

$x_1 = \omega^i + \omega^{i-1} + \gamma$ and $y_1 = \omega^{i-1} \cdot 4 + \gamma$ for some $\gamma < \omega^i$. Then $A^{<x_1} \equiv_{n-1} B^{<y_1}$ by Lemma 2.6(i). Furthermore, $A^{>x_1} \cong B^{>y_1} \cong \omega^i + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot 4$ and hence $A^{>x_1} \equiv_n B^{>y_1}$.

Otherwise x_1 is an element of the last segment $\omega^{i-1} \cdot 3 + \omega^{i-2} \cdot 3 + \ldots + \omega^{i-r} \cdot 4$ of A and y_1 is the corresponding point in the last segment of B. Then $A^{>x_1} \cong B^{>y_1}$, so $A^{>x_1} \equiv_{n-1} B^{>y_1}$. In addition, $A^{<x_1} \cong \omega^i \cdot 2 + \gamma$ and $B^{<y_1} \cong \omega^i + \gamma$, where $\gamma < \omega^i$. By Theorem 1.4(i), $A^{<x_1} \equiv_{n-1} B^{<y_1}$.

Finally suppose that $0 \leq r \leq i < \frac{n-1}{2}$. Player II can play so that one of the following occurs:

$x_1 = y_1 < \omega^i(2^{n-1-2i} + 1)$, giving $A^{<x_1} \equiv_{n-1} B^{<y_1}$, and $A^{>x_1} \cong \omega^i \cdot q_1 + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot 4$ and $B^{>y_1} \cong \omega^i \cdot q_2 + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot 4$ where $q_1 \geq 2^{n-1-2i}$ and $q_2 \geq 2^{n-1-2i} - 1$, in which case $A^{>x_1} \equiv_{n-1} B^{>y_1}$ by induction hypothesis.

$x_1 = \omega^i(2^{n-1-2i} + 1) + \gamma$ and $y_1 = \omega^i \cdot 2^{n-1-2i} + \gamma$ for some γ, and by Lemma 2.3(i), $A^{<x_1} \equiv_{n-1} B^{<y_1}$, and $A^{>x_1} \cong B^{>y_1}$ so $A^{>x_1} \equiv_{n-1} B^{>y_1}$.

(ii) Let $A = \omega^i \cdot 2^{n-2i} + \omega^{i-1} \cdot a_{i-1} + \ldots + \omega \cdot a_1 + a_0$ and $B = \omega^i(2^{n-2i} - 1) + \omega^{i-1} \cdot b_{i-1} + \ldots + \omega \cdot b_1 + b_0$. Player I chooses $x_1 < x_2 < \ldots < x_{n-2i}$ where $x_j = \omega^i(2^{n-2i} - 2^{n-j-2i})$. By Lemma 2.2 applied successively to n, $n-1$, …, $2i+1$ we may suppose that II's moves are $y_1 < y_2 < \ldots < y_{n-2i}$ where $y_j = \omega^i \cdot t_j$ for some t_j. If $t_1 < 2^{n-1-2i}$ then by Lemma 2.3(ii) I wins by playing to the left of x_1, y_1 in the remaining $n-1$ moves, since $A^{<x_1} \cong \omega^i \cdot 2^{n-1-2i} \not\equiv_{n-1} \omega^i \cdot l \cong B^{<y_1}$, where $l < 2^{n-1-2i}$. Similarly, if any interval (y_j, y_{j+1}) is shorter than the corresponding interval (x_j, x_{j+1}), then I wins there in the remaining $n-j-1$ moves. So we may suppose that $t_1 \geq 2^{n-1-2i}$ and $t_{j+1} - t_j \geq 2^{n-j-1-2i}$, so as $\sum_{j=1}^{n-2i} 2^{n-j-2i} = 2^{n-2i} - 1$, in fact $y_j = x_j$ for each j.

In the remaining $2i$ moves, I plays on the right, and as $A^{>x_{n-2i}} \cong \omega^i + \omega^{i-1} \cdot a_{i-1} + \ldots + \omega \cdot a_1 + a_0$ and $B^{>y_{n-2i}} \cong \omega^{i-1} \cdot b_{i-1} + \ldots + \omega \cdot b_1 + b_0$ (except that if $i = 1$, $B^{>y_{n-2i}} \cong b_0 - 1$) we just have to see that $\omega^i + \omega^{i-1} \cdot a_{i-1} + \ldots + \omega \cdot a_1 + a_0 \not\equiv_{2i} \omega^{i-1} \cdot b_{i-1} + \ldots + \omega \cdot b_1 + b_0'$ (where $b_0' = b_0$ if $i > 1$ and it equals $b_0 - 1$ if $i = 1$), and to make the induction work, we actually show that for any ordinal $\alpha > 0$, $\omega^i \cdot \alpha + \omega^{i-1} \cdot a_{i-1} + \ldots + \omega \cdot a_1 + a_0 \not\equiv_{2i} \omega^{i-1} \cdot b_{i-1} + \ldots + \omega \cdot b_1 + b_0'$. The basis case $i = 1$ says that $\omega \cdot \alpha + a_0 \not\equiv_2 b_0 - 1$ which holds since $b_0 \leq 3$: if $a_0 \neq 0$ then $\omega \cdot \alpha + a_0 \equiv_2 3 \not\equiv_2 b_0 - 1$ as $b_0 \leq 3$, and if $a_0 = 0$, $\omega \cdot \alpha + a_0 \not\equiv_2 b_0 - 1$ is immediate.

Assuming the result for i, we describe a winning strategy for player I in the $(2i+2)$-move game on $A_{i+1} = \omega^{i+1} \cdot \alpha + \omega^i \cdot a_i + \ldots + \omega \cdot a_1 + a_0$ and $B_{i+1} = \omega^i \cdot b_i + \ldots + \omega \cdot b_1 + b_0'$. On his first two moves, player I plays $x_1 = \omega^i \cdot 2$ and $x_2 = \omega^i \cdot 3$. Let y_1, y_2 be player II's responses. By Lemma 2.2 we suppose that $y_1 = \omega^i \cdot t_1$ and $y_2 = \omega^i \cdot t_2$. Then $0 < t_1 < t_2 \leq 3$ so $t_1 \leq 2$. If $t_1 = 1$ then $A_{i+1}^{<x_1} \cong \omega^i \cdot 2$ and $B_{i+1}^{<y_1} \cong \omega^i$. By Theorem 1.4(iii), $A_{i+1}^{<x_1} \not\equiv_{2i+1} B_{i+1}^{<y_1}$ so I wins

in the remaining $2i+1$ moves. Otherwise, $t_1 = 2$ and $t_2 = 3$ (so actually $b_i = 3$). Therefore $A_i = A_{i+1}^{>x_2} \cong \omega^{i+1} \cdot \alpha + \omega^i \cdot a_i + \omega^{i-1} \cdot a_{i-1} + \ldots + \omega \cdot a_1 + a_0$ and $B_i = B_{i+1}^{>y_2} \cong \omega^{i-1} \cdot b_{i-1} + \ldots + \omega \cdot b_1 + b_0'$, noting that in the special case where $i = 1$, since y_2 is the first point of the finite block at the end, the previous value of b_0' which equalled b_0 has decreased by 1, to the new (and correct) value of b_0'. Since A_i may be written in the form $\omega^i(\omega \cdot \alpha + a_i) + \omega^{i-1} \cdot a_{i-1} + \ldots + \omega \cdot a_1 + a_0$, we may appeal to the induction hypothesis to see that this is not $2i$-equivalent to B_i, giving the induction step. ■

The next lemma is similar to the previous one, though proved without using induction.

Lemma 2.11. *If α is an ordinal of the form $\omega^j \cdot \gamma$ where $\gamma \geq 1$, and $\alpha \equiv_n \beta$, $1 \leq i < \frac{n}{2} - 1$, $0 < r \leq i < j \leq \frac{n}{2}$, and $m \geq 4$, then*

(i) $\alpha + \omega^i(2^{n-2i} - 4) + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m$
$\equiv_n \beta + \omega^i(2^{n-2i} - 5) + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m,$

(ii) *if $b_k \leq 3$ for all $k < i$, or for some $l < i$, $b_k \leq 3$ for all k such that $l \leq k < i$ and $b_l \leq 2$, then* $\alpha + \omega^i(2^{n-2i} - 4) + \omega^{i-1} \cdot a_{i-1} + \ldots + \omega \cdot a_1 + a_0$
$\not\equiv_n \beta + \omega^i(2^{n-2i} - 5) + \omega^{i-1} \cdot b_{i-1} + \ldots + \omega \cdot b_1 + b_0.$

Proof. (i) Let $A = \alpha + \omega^i(2^{n-2i} - 4) + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m$ and $B = \beta + \omega^i(2^{n-2i} - 5) + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m$.

First suppose that $i = \frac{n-3}{2}$ (so that n is odd). Here $2^{n-2i} = 8$, so $A = \alpha + \omega^i \cdot 4 + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot m$ and $B = \beta + \omega^i \cdot 3 + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot m$. Player II may play so that one of the following holds:

$x_1 \in \alpha$ and $y_1 \in \beta$ correspond under a winning strategy for II in $\alpha \equiv_n \beta$,

x_1 and y_1 are corresponding points of the r_0th and r_1th copies of ω^i where $r_0 = r_1 = 0$ or $r_0 = r_1 + 1 \geq 2$,

x_1 and y_1 are corresponding points of the r_0th copy of ω^i in the $\omega^i \cdot 4$ section of A and of the r_1th copy of ω^i in β where $r_0 = 1$ and $r_1 = 4$,

x_1 and y_1 are corresponding points of $\omega^{i-1} \cdot 3 + \ldots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m$.

To show that this works, we need to verify the following $(n-1)$-equivalences:

$\omega^i \cdot 4 + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot m \equiv_{n-1} \omega^i \cdot 3 + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot m$, which follows from Lemma 2.10(i), since $0 < r \leq i < \frac{n-1}{2}$, and since $n - 1 - 2i = 2$,

$\alpha + \omega^i \cdot 2 \equiv_{n-1} \beta + \omega^i$, which follows from Theorem 1.4(i),

$\alpha + \omega^i \equiv_{n-1} \omega^i \cdot 4$, which follows from Lemma 2.6(i), and

$\omega^i \cdot 3 + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot m \equiv_{n-1} \beta + \omega^i \cdot 3 + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot m$, which holds since $\beta + \omega^i \cdot 3 + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot m$

$\equiv_{n-1} \omega^i \cdot 4 + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot m$ by Lemma 2.6(i)

$\equiv_{n-1} \omega^i \cdot 3 + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r} \cdot m$ by Lemma 2.10(i),

$\alpha + \omega^i \cdot 4 \equiv_{n-1} \beta + \omega^i \cdot 3$, which follows from Lemma 2.6(i).

Next consider the case where $i < \frac{n-3}{2}$. Then $n - 1 - 2i > 2$ so $2^{n-1-2i} - 4 \geq 4$. We subdivide A and B as follows:

$A = \alpha + \omega^i(2^{n-1-2i} - 4) + \omega^i + \omega^i(2^{n-1-2i} - 1) + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m,$

$B = \alpha + \omega^i(2^{n-1-2i} - 4) \qquad + \omega^i(2^{n-1-2i} - 1) + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m.$

Then II can play so that one of the following holds:

$x_1 \in \alpha$ and $y_1 \in \beta$ correspond under a winning strategy for II in $\alpha \equiv_n \beta$,

x_1 and y_1 are corresponding points of the $\omega^i(2^{n-1-2i} - 4)$, $\omega^i(2^{n-1-2i} - 1)$, or $\omega^{i-1} \cdot 3 + \ldots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m$ sections,

x_1 lies in the middle ω^i section of A, and y_1 is the corresponding point of the first ω^i of the $\omega^i(2^{n-1-2i} - 1)$ section of B.

To see that this works, we need to verify the following $(n-1)$-equivalences:

$\omega^i \cdot 2^{n-1-2i} + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m \equiv_{n-1} \omega^i(2^{n-1-2i} - 1) + \omega^{i-1} \cdot 3 + \ldots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m$, which follows from Lemma 2.10(i),

$\alpha + \omega^i(2^{n-1-2i} - 3) \equiv_{n-1} \beta + \omega^i(2^{n-1-2i} - 4)$, which follows from Lemma 2.9(i).

(ii) Let $A = \alpha + \omega^i(2^{n-2i} - 4) + \omega^{i-1} \cdot a_{i-1} + \ldots + \omega \cdot a_1 + a_0$ and $B = \beta + \omega^{i-1}(2^{n-2i} - 5) + \omega^{i-1} \cdot b_{i-1} + \ldots + \omega \cdot b_1 + b_0$. On his first move, player I plays $x_1 = \alpha + \omega^i(2^{n-1-2i} - 4)$. By Lemma 2.2 we may suppose that II plays a multiple y_1 of ω^i. If $y_1 \geq \beta + \omega^i(2^{n-1-2i} - 4)$ then $A^{>x_1} \cong \omega^i \cdot 2^{n-1-2i} + \omega^{i-1} \cdot a_{i-1} + \ldots + \omega \cdot a_1 + a_0$ and $B^{>y_1} \cong \omega^i \cdot q_1 + \omega^{i-1} \cdot b_{i-1} + \ldots + \omega \cdot b_1 + b_0$ for some $q_1 < 2^{n-1-2i}$, and I wins by Lemma 2.10(ii).

We suppose therefore that $y_1 < \beta + \omega^i(2^{n-1-2i} - 4)$. If $y_1 = \beta + \omega^i \cdot q_2$ for some $q_2 < 2^{n-1-2i} - 4$, then I wins by Lemma 2.9(ii). Otherwise, $y_1 < \beta$. Next I plays $y_2 > y_1$ so that $(y_1, y_2) \cong \omega^{i+1}$, and whatever x_2 II plays, $(x_1, x_2) \cong \gamma_1 + \omega^r$ for some ordinal γ_1, where $r \leq i$. Thus, provided $i + 1 \leq \frac{n-2}{2}$, I wins by using Lemma 2.1. If this fails, then $i + 1 > \frac{n-2}{2}$, and since $i < \frac{n}{2} - 1$, the only possibility remaining that we need to cover is where $i = \frac{n-3}{2}$ (so in particular, n is odd). Since $n - 2i = 3$, in this case, $A = \alpha + \omega^i \cdot 4 + \omega^{i-1} \cdot a_{i-1} + \ldots + \omega \cdot a_1 + a_0$ and $B = \beta + \omega^i \cdot 3 + \omega^{i-1} \cdot b_{i-1} + \ldots + \omega \cdot b_1 + b_0$.

In this instance, player I plays $x_1 = \alpha + \omega^i \cdot 2$ and unless II plays in β, as we have seen, I can win as before, so suppose that $y_1 < \beta$. Now player I plays y_2 so that $(y_1, y_2) \cong \omega^{i+1}$. By Lemma 2.2, since $i < \frac{n-1}{2}$, we may suppose that II now plays a multiple x_2 of ω^i. If $x_2 = \alpha + \omega^i \cdot 3$, then $(x_1, x_2) \cong \omega^i$ and $(y_1, y_2) \cong \omega^{i+1}$, and I wins by Theorem 1.4(iii) since $2i + 1 = n - 2$. Otherwise, $x_2 = \alpha + \omega^i \cdot 4$, and now player I plays y_3 so that $(y_2, y_3) \cong \omega^i$, and as $i < \frac{n-2}{2}$, we may suppose that II plays a multiple of ω^i. But since $x_2 = \alpha + \omega^i \cdot 4$ this is impossible, and so I wins. ■

We are now ready to state the conditions which feature in the main result of this section. Recall that t stands for the integer part of $\frac{n}{2}$.

One of the clauses refers to the notion of 'n-optimality' of a coefficient; this is only required when this coefficient is 3, and we find it easiest to give an explicit definition. Namely, 3 is an *n-optimal coefficient for ω^k in $\alpha = \omega^i \cdot a_i + \ldots + a_0 \in \Omega'_n$ if $a_i \neq 0$, $k < i$, $a_k = 3$* and either $i < \frac{n}{2}$ and $a_i = 2^{n-2i}$, $a_{i-1} = \ldots = a_k = 3$, or for some $j < i$ such that $k < j$, $a_j = 2^{n-2j} - 4$, $a_{j-1} = \ldots = a_k = 3$. (The intuition, and an alternative definition as in [3], is that increasing the coefficient of ω^k from 3 to 4 results in an ordinal which does *not* lie in Ω'_n—see Lemmas 2.10 and 2.11, but this seems harder to work with in practice.)

For any $n \geq 0$, we let Ω'_n be the set of ordinals of the form

$$\omega^t \cdot a_t + \omega^{t-1} \cdot a_{t-1} + \omega^{t-2} \cdot a_{t-2} + \ldots + \omega \cdot a_1 + a_0$$

such that

$$a_i \leq \begin{cases} 2^{n-2i} & & (1) \\ 2^{n-2i} - 4 & \text{if } a_j \neq 0 \text{ for some } j > i & (2) \\ 3 & \text{if } i < \frac{n}{2} - 1 \text{ and } a_{i+1} = 2^{n-2(i+1)} \text{ or} & (3) \\ & \text{if } a_{i+1} = 2^{n-2(i+1)} - 4 \text{ and } a_j \neq 0 \text{ for some } j > i+1 \text{ or} & (4) \\ & \text{if } a_{i+1} = 3 \text{ and 3 is an } n\text{-optimal coefficient for } \omega^{i+1} & (5) \\ 2^n - 1 & \text{if } i = 0 \text{ and } a_j = 0 \text{ for all } j > 0 & (6) \end{cases}$$

In Lemma 2.12 we shall establish 'optimality'.

Lemma 2.12. *Let $\alpha = \omega^i \cdot a_i + \omega^{i-1} \cdot a_{i-1} + \ldots + \omega \cdot a_1 + a_0$ and $\beta = \omega^i \cdot b_i + \omega^{i-1} \cdot b_{i-1} + \ldots + \omega \cdot b_1 + b_0$ where $\alpha \in \Omega'_n$. If $\alpha > \beta$, then $\alpha \not\equiv_n \beta$.*

Proof. We must show that player I has a winning strategy in the n-move game on α and β. Since $\alpha \in \Omega'_n$, $a_j \leq 2^{n-2j}$ for each j.

If $a_i = 0$ then as $\beta < \alpha$, also $b_i = 0$, so these terms could be omitted. We may therefore assume that $a_i \neq 0$. If α is finite, then the result follows from Lemma 1.3, so from now on we assume that α is infinite, that is, $i > 0$.

Let j be the largest number such that $a_0 = a_1 = \ldots = a_{j-1} = 0$ (so that $j \leq i$). Unless also $b_0 = b_1 = \ldots = b_{j-1} = 0$, I can win, as otherwise if $r < j$ is the least such that $b_r \neq 0$ then we can write α as $\omega^j \cdot \gamma_0$ and β as $\gamma_1 + \omega^r$, so by Lemma 2.1, $\alpha \not\equiv_n \beta$. Similarly in reverse. Hence we may suppose that the last non-zero terms of the expressions for α and β occur at the same point.

We now fix $j \leq i$ as the first point for which $a_j \neq b_j$. Since $\alpha > \beta$, $a_j > b_j$.

Case 1: $a_{j-1} = a_{j-2} = \ldots = a_0 = 0$. Then by the above, also $b_{j-1} = b_{j-2} = \ldots = b_0 = 0$. If $j = i$ then $\alpha \not\equiv_n \beta$ by Lemma 2.3(ii). If $j < i$ then by clause (2), $a_j \leq 2^{n-2j} - 4$. Since $a_j > 0$, it follows that $n - 2j > 2$, so that $\frac{n-3}{2} \geq j$. We may therefore appeal to Lemma 2.9(ii) to see that $\alpha \not\equiv_n \beta$. (Strictly speaking, we have to deal with the case where $a_j < 2^{n-2j} - 4$, but if $\omega^i \cdot a_i + \ldots + \omega^{j+1} \cdot a_{j+1} + \omega^j \cdot a_j \equiv_n \omega^i \cdot a_i + \ldots + \omega^{j+1} \cdot a_{j+1} + \omega^j \cdot b_j$ then also $\omega^i \cdot a_i + \ldots + \omega^{j+1} \cdot a_{j+1} + \omega^j (2^{n-2j} - 4) \equiv_n \omega^i \cdot a_i + \ldots + \omega^{j+1} \cdot a_{j+1} + \omega^j (b_j + 2^{n-2j} - 4 - a_j)$ to which the lemma applies.)

Case 2: For some $r \leq \frac{n}{2} - 2$, $b_r < a_r$ and $b_r < 2^{n-2-2r}$ or $a_r < b_r$ and $a_r < 2^{n-2-2r}$.

We suppose that $b_r < a_r$ and $b_r < 2^{n-2-2r}$.

On his first move, player I chooses $x_1 = \omega^i \cdot a_i + \ldots + \omega^{r+1} \cdot a_{r+1} \in \alpha$. Since $r + 1 < \frac{n}{2}$, by Lemma 2.2 we may suppose that y_1 is a multiple of ω^{r+1}.

If $y_1 < \omega^i \cdot b_i + \ldots + \omega^{r+1} \cdot b_{r+1}$ then I plays $y_2 = \omega^i \cdot b_i + \ldots + \omega^{r+1} \cdot b_{r+1}$, and now (y_1, y_2) is a multiple of ω^{r+1}. Whatever x_2 is played by II, $(x_1, x_2) \cong \gamma + \omega^s$ for some $s \leq r$, and so as $s < r + 1 \leq \frac{n-2}{2}$, I wins by appeal to Lemma 2.1. If however $y_1 = \omega^i \cdot b_i + \ldots + \omega^{r+1} \cdot b_{r+1}$, I instead plays $x_2 = \omega^i \cdot a_i + \ldots + \omega^{r+1} \cdot a_{r+1} + \omega^r (b_r + 1)$, provided this is $< \alpha$ (and if it equals α, he plays $x_2 = \omega^i \cdot a_i + \ldots + \omega^{r+1} \cdot a_{r+1} + \omega^r \cdot b_r$; this happens if $a_r = b_r + 1$ and all a_s for $s < r$ are zero). Since $r < \frac{n-1}{2}$, Lemma 2.2 allows us to assume that II plays a multiple y_2 of ω^r. Then $(x_1, x_2) \cong \omega^r (b_r + 1)$ (or $\omega^r \cdot b_r$ in the second case) and $(y_1, y_2) \cong \omega^r \cdot t_0$ where $t_0 \leq b_r$ (or $t_0 < b_r$ in the second case). It follows by Lemma 2.3(ii) that $(x_1, x_2) \not\equiv_{n-2} (y_1, y_2)$, and so I wins in the remaining $n - 2$ moves.

Cases 3, 4, 5, and 6 cover all instances in which $j = i$.

Case 3: $j = i < \frac{n}{2}$ and $b_i < 2^{n-1-2i}$.

Player I chooses $x_1 = \omega^i (b_i + 1)$. Let $y_1 \in \beta$ be II's reply. By Lemma 2.2 we may suppose that $y_1 = \omega^i \cdot t_1$ for some t_1 and then $\beta^{<y_1} \not\equiv_{n-1} \alpha^{<x_1}$ by Lemma 2.3(ii).

Case 4: $j = i < \frac{n}{2}$ and $2^{n-1-2i} \leq b_i \leq 2^{n-2i} - 2$.

Player I plays $x_1 < x_2 < \ldots$ as far as possible so that $x_1 = \omega^i \cdot 2^{n-1-2i}$, and $(x_k, x_{k+1}) \cong \omega^i \cdot 2^{n-k-1-2i}$. Since $a_i > 2^{n-1-2i}$, x_1 exists. Now assume that $x_1, x_2, \ldots, x_k$ have been chosen fulfilling these conditions. Then $x_k = \omega^i (2^{n-1-2i} + 2^{n-2-2i} + \ldots + 2^{n-k-2i}) = \omega^i (2^{n-2i} - 2^{n-k-2i})$. Thus $a_i \geq 2^{n-2i} - 2^{n-k-2i}$, and if $a_i \geq 2^{n-2i} - 2^{n-k-1-2i}$ then x_{k+1} can be chosen as desired. Otherwise, if $x_k = \omega^i \cdot a_i$, then we stop with $r = k$, and if $x_k < \omega^i \cdot a_i$ we let $x_{k+1} = \omega^i \cdot a_i$ and $r = k+1$. Then $(x_k, x_{k+1}) \cong \omega^i (a_i - (2^{n-2i} - 2^{n-k-2i})) < \omega^i ((2^{n-2i} - 2^{n-k-1-2i}) - (2^{n-2i} - 2^{n-k-2i})) = \omega^i (2^{n-k-2i} - 2^{n-k-1-2i})$. Note that $r \leq n - 2i$, and there are $n - r \geq 2i$ moves remaining.

Let II's moves in β be $y_1 < y_2 < \ldots < y_r$. By Lemma 2.2 applied to $n, n-1, \ldots, n-(r-1)$ we may suppose that $y_k = \omega^i \cdot t_k$ for some t_k. If $t_1 < 2^{n-1-2i}$ then I wins by Lemma 2.3(ii), and similarly if any of the intervals between successive y_ks is less than the corresponding intervals between the x_ks. So we suppose that

$t_1 \geq 2^{n-1-2i}$, $t_2 \geq 2^{n-2-2i}$, and so on, which shows that $b_i \geq a_i$ after all, contrary to supposition.

Case 5: $j = i < \frac{n}{2}$ and $b_i = 2^{n-2i} - 1$.

It follows from the hypotheses that $a_i = 2^{n-2i}$.

To streamline consideration of the cases which can arise, we note that by Lemma 2.10(ii), if $b_l \leq 3$ for all $l \leq i-1$, or if there is $k \leq i-1$ such that $b_l \leq 3$ for all l with $k \leq l \leq i-1$ and $b_k \leq 2$, then $\omega^i \cdot 2^{n-2i} + \omega^{i-1} \cdot a_{i-1} + \ldots + \omega \cdot a_1 + a_0 \not\equiv_n \omega^i(2^{n-2i} - 1) + \omega^{i-1} \cdot b_{i-1} + \ldots + \omega \cdot b_1 + b_0$. Thus we may assume that for some $l \leq i-1$, $b_{i-1} = b_{i-2} = \ldots = b_{l+1} = 3$ and $b_l > 3$. Let k be greatest such that $l \leq k \leq i-1$ and $a_k \neq b_k$. This exists, since if not, then $a_k = b_k$ for all k in this range, which implies that $a_l > 3$, violating clause (5) (or clause (3) if $l = i-1$). Hence, if $k < m \leq i-1$, $a_m = b_m = 3$, so by the definition of 'n-optimal coefficient', and clause (5), $a_k \leq 3$, and hence $a_k < b_k$.

Case 5A: For the k just defined, $k \leq \frac{n}{2} - 2$.

Since $a_k \leq 3 < 4 = 2^{n-2-n+4} \leq 2^{n-2-2k}$, the result follows from Case 2.

Case 5B: $k > \frac{n}{2} - 2$.

It follows that $k \geq \frac{n-3}{2}$, so as $k < i < \frac{n}{2}$, n must be odd, $i = \frac{n-1}{2}$, and $k = i-1$.

Then $\alpha = \omega^i \cdot 2 + \omega^{i-1} \cdot a_{i-1} + \ldots + a_0$ and $\beta = \omega^i + \omega^{i-1} \cdot b_{i-1} + \ldots + b_0$, $a_{i-1} \leq 3$ and $a_{i-1} < b_{i-1}$. Let m be least if any such that $a_{i-1} = \ldots = a_m = 3$. Then if m exists, either $m = 0$ or $a_{m-1} < 3$, and if it does not exist, then $a_{i-1} < 3$. Now I plays $x_1 = \omega^i \cdot 2$, and by appealing to Lemma 2.2, II must play $y_1 = \omega^i$. Next I plays $y_2 = \omega^i + \omega^{i-1} \cdot 2$ and by Lemma 2.2 we see that player II must play $x_2 = \omega^i \cdot 2 + \omega^{i-1} \cdot t_0$, where $0 < t_0 \leq a_{i-1}$. If $t_0 = 1$, then $(x_1, x_2) \cong \omega^{i-1} \not\equiv_{2i-1} \omega^{i-1} \cdot 2 \cong (y_1, y_2)$ by Theorem 1.4(iii), and $2i - 1 = n - 2$, so I wins. Hence we suppose that $t_0 = 2$. It follows that $\beta > \omega^i + \omega^{i-1} \cdot 3$, as if $\beta = \omega^i + \omega^{i-1} \cdot 3$ then $b_{i-1} = 3$ and $b_m = 0$ for all $m < i-1$ and therefore $a_{i-2} = 2$ and $a_m = 0$ for all $m < i-1$. Hence I can play $y_3 = \omega^i + \omega^{i-1} \cdot 3$. By Lemma 2.2, II must respond with $x_3 = \omega^i \cdot 2 + \omega^{i-1} \cdot t_1$ where $t_0 < t_1 \leq a_{i-1}$. Thus $t_1 = 3$ (so that $a_{i-1} = 3$, and m is defined, and $b_{i-1} > 3$). On subsequent moves, player I plays in β in the section $(\omega^i + \omega^{i-1} \cdot 3, \omega^i + \omega^{i-1} \cdot 4)$ at intervals of ω^{i-2}, and player II is unable to respond at all stages, and I wins.

Case 6: $j = i = \frac{n}{2}$.

Thus n is even. By clause (2), $a_{i-1} \leq 2^2 - 4 = 0$, so $\alpha = \omega^i + \omega^{i-2} \cdot a_{i-2} + \ldots + a_0$ and $\beta = \omega^{i-1} \cdot b_{i-1} + \ldots + b_0$. By clause (4), $a_{i-2} \leq 3$.

If $b_{i-1} = 0$, player I plays $x_1 = \omega^i$ and by Lemma 2.2, player II must play a multiple of ω^{i-1}, but this is impossible, and so he loses. From now on we therefore suppose that $b_{i-1} > 0$.

Look at the first $k \leq i-2$, if any, such that $a_k \neq b_k$ or $a_k = b_k \neq 3$. By clause (4) or (5), $a_k \leq 3$. Hence if $a_k \neq b_k$, we may use Case 2 to deduce that $\alpha \not\equiv_n \beta$. If $a_k = b_k \neq 3$, then as $a_k \leq 3$, actually $a_k = b_k < 3$, and by maximality of k, $b_{i-2} = b_{i-3} = \ldots = b_{k+1} = 3$. Player I plays $y_1 = \omega^{i-1}(b_{i-1} - 1)$. Then by Lemma 2.2, II must play a multiple of ω^{i-1}. If this is $x_1 = \omega^i$, then I plays $y_2 = \omega^{i-1} \cdot b_{i-1}$ and $(x_1, x_2) \cong \gamma + \omega^r$ for some ordinal γ, and $r \leq i-2$, so as $r < i - 1 \leq \frac{n-2}{2}$, I wins in the remaining $n-2$ moves by Lemma 2.1. Now assume that II plays $x_1 = \omega^{i-1} \cdot t_0$ for some finite t_0. From now on, I plays $x_2, x_3, \ldots$ as long as necessary so that $(x_s, x_{s+1}) \cong \omega^{i-1}$. Let $y_2, y_3, \ldots$ be II's replies. If $y_2 = \gamma + \omega^r$ for some $r \leq i-2$ then I wins by Lemma 2.1. So we assume that $y_2 = \omega^{i-1} \cdot b_{i-1}$. If for some $s \geq 2$, $(y_s, y_{s+1}) \not\equiv_{n-s-1} \omega^{i-1}$, then I can win on (x_s, x_{s+1}) and (y_s, y_{s+1}) in the remaining $n - s - 1$ moves, so we show that this happens for some $s \leq n-1$. Suppose for a contradiction therefore that for each $s \leq n-1$, $(y_s, y_{s+1}) \equiv_{n-s-1} \omega^{i-1}$.

Let $(y_s, y_{s+1}) \cong \gamma + \omega^r$. Then if $r \leq \frac{n-s-3}{2}$, by Lemma 2.1, $(y_s, y_{s+1}) \not\equiv_{n-s-1} \omega^{i-1}$. Hence $r > \frac{n-s-3}{2}$. If s is even this tells us that $r \geq \frac{n-s-2}{2}$, and if s is odd, that $r \geq \frac{n-s-1}{2}$. We deduce that $y_3 = y_2 + \omega^{i-2} \cdot t_0$ and $y_4 = y_2 + \omega^{i-2} \cdot t_1$ for some $t_1 > t_0 > 0$. By Theorem 1.4(iii), $\omega^{i-1} \not\equiv_{n-3} \omega^{i-2}$, so as $(x_2, x_3) \cong \omega^{i-1}$ and $(x_2, x_3) \not\equiv_{n-3} (y_2, y_3)$, it follows that $(y_2, y_3) \not\cong \omega^{i-2}$ and hence $t_0 > 1$. Since $y_4 \leq \omega^{i-1} \cdot b_{i-1} + \omega^{i-2} \cdot b_{i-2}$ and $b_{i-2} = 3$ we deduce that $t_0 = 2$, $t_1 = 3$, and $y_4 = \omega^{i-1} \cdot b_{i-1} + \omega^{i-2} \cdot b_{i-2}$. Repeating this argument, we find that $y_6 = \omega^{i-1} \cdot b_{i-1} + \omega^{i-2} \cdot b_{i-2} + \omega^{i-3} \cdot b_{i-3}, \ldots$, and when we reach the term in ω^k, the corresponding t_0 is forced to be 1 since $b_k \leq 2$, giving a contradiction.

Finally, suppose that there is no such k. It follows from clause (5) that for every $k \leq i-2$, $a_k = b_k = 3$. In the play described above, player II's moves must continue to the constant term. He has now played $2(i-1)+1 = n-1$ moves and ends by playing the final point of β, so I wins on the last move.

We now consider cases in which $j < i$, and subdivide in a a similar way to Cases 3-6. Since we suppose that Case 1 does not apply, $a_k \neq 0$ for some $k < j$. Note further that if $j = \frac{n}{2} - 1$ then $i = \frac{n}{2}$, so that $a_j = 0$, contrary to $b_j < a_j$. Hence $j < \frac{n}{2} - 1$. By clause (2), $a_j \leq 2^{n-2j} - 4$.

Case 7: $j < i$, $a_k \neq 0$ for some $k < j$, and $b_j < 2^{n-1-2j} - 4$.

Note that it follows from this that $n - 1 - 2j > 2$, so $j < \frac{n-3}{2}$.

Player I chooses $x_1 = \omega^i \cdot a_i + \ldots + \omega^{j+1} \cdot a_{j+1} + \omega^j (b_j + 1)$. Then II's response y_1 must be a multiple of ω^j (using Lemma 2.2, since $j < \frac{n}{2}$). If $y_1 < \omega^i \cdot a_i + \ldots + \omega^{j+1} \cdot a_{j+1}$ then I plays $y_2 = \omega^i \cdot a_i + \ldots + \omega^{j+1} \cdot a_{j+1}$. Whatever II's reply x_2 is, $x_2 = \gamma + \omega^r$ for some $r \leq j$, so as $r < j + 1 \leq \frac{n-2}{2}$, I wins using Lemma 2.1. If $y_1 = \omega^i \cdot a_i + \ldots + \omega^{j+1} \cdot a_{j+1}$ then I plays $x_2 < x_1$ so that $(x_2, x_1) \cong \omega^j$. Whatever y_2 is chosen by II, (y_2, y_1) is a multiple of ω^{j+1}, and so by Theorem 1.4(iii), $(x_2, x_1) \not\equiv_{2j+1} (y_2, y_1)$, so as $2j + 1 \leq n - 2$, I wins. Otherwise, $y_1 = \omega^i \cdot a_i + \ldots \omega^{j+1} \cdot a_{j+1} + \omega^j \cdot t_1$ for some t_1 with $0 < t_1 \leq b_j$. By Lemma 2.9(ii) (recalling the remark at the end of Case 1), $\alpha^{<x_1} \not\equiv_{n-1} \beta^{<y_1}$, so I wins.

Case 8A: $j < i$, $a_k \neq 0$ for some $k < j$, $j < \frac{n-3}{2}$, and $2^{n-1-2j} - 4 \leq b_j < 2^{n-2j} - 5$.

We follow a similar strategy to Case 4.

Player I chooses $x_1 < x_2 < \ldots$ as far as possible so that $x_l = \omega^i \cdot a_i + \ldots + \omega^{j+1} \cdot a_{j+1} + \omega^j \cdot t_l$, where $t_l = 2^{n-2j} - 2^{n-l-2j} - 4$, and at the first point where this is impossible, that is, $2^{n-2j} - 2^{n-(l-1)-2j} - 4 \leq a_j < 2^{n-2j} - 2^{n-l-2j} - 4$, we stop at $r = l - 1$ if $t_{l-1} = a_j$ and let $t_l = a_j$ otherwise (and then stop with $r = l$). In all cases, $t_r - t_{r-1} \leq 2^{n-r-2j}$. Let $y_1 < y_2 \ldots$ be II's responses. As in Case 7, we may appeal to Lemma 2.2 to see that each y_r may be assumed to be a multiple of ω^j. Furthermore, we may suppose that there are $t'_1 < t'_2 < \ldots$ such that $t'_1 \geq 1$ and $y_r = \omega^i \cdot a_i + \ldots + \omega^{j+1} \cdot a_{j+1} + \omega^j \cdot t'_r$. We mainly have to justify this for y_1. The only other options are that $y'_1 \leq \omega^i \cdot a_i + \ldots + \omega^{j+1} \cdot a_{j+1}$, when the argument of Case 7 applies, in the case of strict inequality appealing to $j < \frac{n-3}{2}$.

Since $b_j < a_j$, $t'_1 < t_1$, or there is l such that $t'_{l+1} - t'_l < t_{l+1} - t_l$. In the first case we appeal to Lemma 2.9(ii) to deduce that $\alpha^{<x_1} \not\equiv_{n-1} \beta^{<y_1}$, and in the latter to Lemma 2.3(ii) to deduce that $(x_l, x_{l+1}) \not\equiv_{n-l-1} (y_l, y_{l+1})$, so in each case, player I wins in the remaining moves.

Case 8B: $j < i$, $a_k \neq 0$ for some $k < j$, $j \geq \frac{n-3}{2}$, and $2^{n-1-2j} - 4 \leq b_j < 2^{n-2j} - 5$.

Since $j < \frac{n}{2} - 1$, in fact $j = \frac{n-3}{2}$ and $i = \frac{n-1}{2}$. By clauses (1) and (3), $a_i \leq 2$, and if $a_i = 2$, then $a_{i-1} \leq 3$, in each case $b_i = a_i$ and $b_{i-1} < a_{i-1}$. If $a_i = 1$, then we follow the proof of Case 8A, noting that $t_1 = 2^{n-2j-1} - 4 = 0$, so that $x_1 = \omega^i$, and Lemma 2.2 ensures that y_1 may be assumed to be a multiple of ω^i too, hence equal to ω^i.

So we concentrate on the case $a_i = 2$, $b_{i-1} < a_{i-1} \leq 3$. Player I plays $x_1 = \omega^i \cdot 2$ and by Lemma 2.2, II must play $y_1 = \omega^i$ or $\omega^i \cdot 2$. If $y_1 = \omega^i$, player I now plays $y_r = \omega^i + \omega^{i-1}(r-1)$ in β as long as necessary, and as in previous proofs, player II is unable to respond for all of the remaining $n-1$ moves. If however $y_1 = \omega^i \cdot 2$, then player I plays $x_2 = \omega^i \cdot 2 + \omega^{i-1}(b_{i-1}+1)$. By Lemma 2.2, II must play a multiple y_2 of ω^{i-1} which is $\leq \omega^i \cdot 2 + \omega^{i-1} \cdot b_{i-1}$. If $y_2 = \omega^i \cdot 2 + \omega^{i-1}$ then I wins on (x_1, x_2) and (y_1, y_2) by Theorem 1.4(iii) using $\omega^{i-1} \not\equiv_{2(i-1)+1} \omega^{i-1} \cdot 2$. Otherwise $y_2 = \omega^i \cdot 2 + \omega^{i-1} \cdot 2$ so $b_{i-1} = 2$ and $a_{i-1} = 3$. Now player I plays $x_3 = \omega^i \cdot 2 + \omega^{i-1} \cdot 3$, and since $i - 1 < \frac{n-2}{2}$, II must play a multiple of ω^{i-1}, which is impossible.

Case 9: $j < i$, $a_k \neq 0$ for some $k < j$, and $2^{n-2j} - 5 \leq b_j$.

Since $b_j < a_j \leq 2^{n-2j} - 4$, it follows that $a_j = 2^{n-2j} - 4$ and $b_j = 2^{n-2j} - 5$. We follow the method of Case 6. Let $k \leq j-1$ be the greatest (if any) such that $a_k \neq b_k$ or $a_k = b_k \neq 3$. As before, it follows that $a_k \leq 3$, and so in the second case $a_k = b_k \leq 2$, and for $k < l \leq j-1$, $a_l = b_l = 3$, so by Lemma 2.11(ii), $\alpha \not\equiv_n \beta$. If $a_k \neq b_k$, we can conclude by appealing to Case 2, since $k < \frac{n}{2} - 2$ and $2^{n-2-2k} \geq 4$, so that $a_k < 2^{n-2-2k}$ (and if $b_k < a_k$, also $b_k < 2^{n-2-2k}$). Thus the fact that the coefficients which control the situation belong to smaller powers of ω than in the previous cases, means that we avoid the two extra cases, corresponding to Cases 5B and 6. ■

Theorem 2.13. *The members of Ω'_n are the minimal representatives of the n-equivalence classes of monochromatic ordinals.*

Proof. We have to show that for any ordinal α, the least ordinal α' which is n-equivalent to α lies in Ω'_n, and also that no two members of Ω'_n are n-equivalent. By Corollary 2.4, α' may be written in the form $\omega^t \cdot a_t + \omega^{t-1} \cdot a_{t-1} + \ldots + \omega \cdot a_1 + a_0$ where $a_i \leq 2^{n-2i}$. Since the truth of the result for $n = 0, 1, 2$ is verified from the lists explicitly given earlier, we assume that $n \geq 3$.

We first establish the numbered properties of α', (1) having already been done.

(2) If there is $j > i$ such that $a_j \neq 0$ and $i \leq \frac{n-3}{2}$, then by Lemma 2.9(i), $a_i \leq 2^{n-2i} - 4$. Since $j \leq \frac{n}{2}$, the only other possibility is that n is even and $i = \frac{n}{2} - 1$, in which case $2^{n-2i} - 4 = 0$, and we appeal to Lemma 2.6(i), since it implies that if $a_{t-1} > 0$ then $\omega^{t-1}(\omega + a_{t-1}) \equiv_n \omega^{t-1} \cdot 4$, which would contradict the minimality of α'.

(3) If $a_{i+1} = 2^{n-2(i+1)}$, then $a_i \leq 3$ because if $a_i \geq 4$ then by Lemma 2.7(i), which may be written in the form $\omega^{i+1} \cdot k + \omega^i \cdot m \equiv_n \omega^{i+1}(2^{n-2(i+1)} - 1) + \omega^i \cdot m$ (assuming that $k \geq 2^{n-2(i+1)}$ and $m \geq 4$), we could reduce the coefficient of ω^{i+1}, contrary to minimality of α'.

(4) If $2^{n-2(i+1)} - 4 = 0$ then $n = 2(i+2)$ so n is even and $i = \frac{n}{2} - 2$. We may use Corollary 2.8(i), since if $a_i \geq 4$ then we could remove the term in ω^t. Otherwise we use Lemma 2.11(i), (replacing i by $i+1$), which tells us that $\omega^j + \omega^{i+1}(2^{n-2(i+1)} - 4) + \omega^i \cdot 4 \equiv_n \omega^j + \omega^{i+1}(2^{n-2(i+1)} - 5) + \omega^i \cdot 4$, so if $a_i \geq 4$, we could reduce α' by decreasing the coefficient of ω^{i+1}.

(5) If $a_{i+1} = 3$ and 3 is an n-optimal coefficient for ω^{i+1}, then by Lemmas 2.10(i) and 2.11(i), and the definition of 'n-optimality', $a_i \leq 3$.

(6) $a_0 \leq 2^n - 1$ by Lemma 1.3 if $a_j = 0$ for all $j > i$.

The converse statement, that any $\alpha = \omega^i \cdot a_i + \omega^{i-1} \cdot a_{i-1} + \ldots + \omega \cdot a_1 + a_0$ fulfilling all clauses is optimal, where $a_i \neq 0$, follows from Lemma 2.12. ■

To illustrate the above rather complicated proof, we consider the following cases: $n = 4, 5$ and 6. We can say that each of these is generated by a finite list of 'maximal' polynomials in ω. (Here 'maximality' is with respect to the partial ordering given by $\sum \omega^i \cdot a_i \succeq \sum \omega^i \cdot b_i$ if for all i, $a_i \geq b_i$.) For $n = 4$ this list is

15, $\omega \cdot 3 + 12$, $\omega \cdot 4 + 3$, $\omega^2 + 3$,

for $n = 5$ the list is

31, $\omega \cdot 7 + 28$, $\omega \cdot 8 + 3$, $\omega^2 \cdot 2 + \omega \cdot 2 + 28$, $\omega^2 \cdot 2 + \omega \cdot 3 + 3$, $\omega^2 + \omega \cdot 3 + 28$, $\omega^2 + \omega \cdot 4 + 3$,

and for $n = 6$ it is

63, $\omega \cdot 15 + 60$, $\omega \cdot 16 + 3$, $\omega^2 \cdot 3 + \omega \cdot 11 + 60$, $\omega^2 \cdot 3 + \omega \cdot 12 + 3$, $\omega^2 \cdot 4 + \omega \cdot 3 + 3$, $\omega^2 \cdot 4 + \omega \cdot 2 + 60$, $\omega^3 + \omega \cdot 2 + 60$, $\omega^3 + \omega \cdot 3 + 3$.

What we mean is that the full list of optimal ordinals is obtained from these 'maximal' ones by allowing the integer coefficients to decrease, so if $\omega^r \cdot a_r + \omega^{r-1} \cdot a_{r-1} + \ldots \omega \cdot a_1 + a_0$ is one of the maximal ones, then the corresponding entries in the full list are those of the form $\omega^r \cdot b_r + \omega^{r-1} \cdot b_{r-1} + \ldots \omega \cdot b_1 + b_0$ where $b_i \leq a_i$ for each i.

Examining the case of $n = 6$ in detail, we see that $t = 3$, and by Corollary 2.4, every ordinal is 6-equivalent to one of the form $\omega^3 \cdot a_3 + \omega^2 \cdot a_2 + \omega \cdot a_1 + a_0$ where $a_3 \leq 1$, $a_2 \leq 4$, $a_1 \leq 16$ and $a_0 \leq 64$. In the first case, $a_3 = 1$, in which case, by (2), $a_2 = 0$, and by applying (4) to $i = 1$, $a_1 \leq 3$. In all cases $a_0 \leq 60$ by (4), and if $a_1 = 3$ then $a_0 \leq 3$ by (5). This is because, by definition, 3 is a 6-optimal coefficient for ω in $\omega^3 + \omega \cdot 3 + a_0$. Now considering the case in which $a_3 = 0$, we look at the various possibilities for a_2. If $a_2 = 4$ then $a_1 \leq 3$ by (3) and if $a_1 = 3$ then $a_0 \leq 3$ by (5). For by definition, 3 is a 6-optimal coefficient for ω in $\omega^2 \cdot 4 + \omega \cdot 3 + a_0$. If however $0 < a_2 \leq 3$ then $a_1 \leq 12$ by clause (2). If $a_1 = 12$ then $a_0 \leq 3$ by clause (4), and if $a_1 \leq 11$ then $a_0 \leq 60$ by (2). Next suppose that $a_3 = a_2 = 0$. Then if $a_1 = 16$, it follows by (3) that $a_0 \leq 3$, and if $0 < a_1 \leq 15$ then $a_0 \leq 60$ by (2). Finally, if $a_3 = a_2 = a_1 = 0$, then $a_0 \leq 63$ by (6). The other cases can be similarly treated.

We conclude this section by remarking that there is a computable function f such that for each n, $f(n)$ lists minimal representatives of the n-equivalence classes of ordinals. To make sense of this, we should encode the ordinals in some standard way; in this case we can just regard the ordinal $\alpha = \omega^k \cdot a_k + \omega^{k-1} \cdot a_{k-1} + \omega^{k-2} \cdot a_{k-2} + \ldots + \omega \cdot a_1 + a_0$ as represented by the finite sequence $(a_k, a_{k-1}, \ldots, a_1, a_0)$ which in turn may be prime power encoded if desired. The function f is then obtained by letting $f(n)$ list (codes for) the members of Ω'_n in increasing order. The fact that f is computable follows from the very explicit definition given of Ω'_n.

3. m-coloured ordinals up to n-equivalence

In this section, we give an analysis of m-coloured ordinals up to n-equivalence. It is a triviality that there is a countable ordinal α such that every m-coloured ordinal $(X, <, F)$ is n-equivalent to some m-coloured ordinal less than α. Namely, for each $(X, <, F)$ we find a suitable countable ordinal by the Löwenheim–Skolem Theorem (which is even *elementarily* equivalent to X), and as there are only finitely many $\equiv_n$-classes, we can just take the maximum of these ordinals. The point however is to find a much smaller, and explicit bound, in the style of [4]. We would like to find a complete and explicit set of representatives as in the monochromatic case in the previous section, but this seems too ambitious at present. Some precise information was given in [4] for 2 moves, but for larger values of n, things get considerably more complicated. We are able to obtain the same *overall* bound as in the monochromatic case, namely, ω^ω. However, this is approached much more rapidly by the individual upper bounds provided by our main theorem as the number of moves n increases, and it seems to us likely that the true value will be considerably lower.

A key tool will be the 'cutting lemma' as given in [4], which applies also in the infinite case, by the same proof as there, and this says the following.

Lemma 3.1. *Let A be an m-coloured linear order and let a and b be elements of A such that $a < b$ satisfying the following conditions:*

(i) *$F(a) = F(b)$,*
(ii) *a and b determine the same n-character, that is, $\langle [A^{<a}]_n, [A^{>a}]_n\rangle = \langle [A^{<b}]_n, [A^{>b}]_n\rangle$,*
(iii) *for every $x \in A$ with $a < x \le b$, there is $y \le a$ of the same colour as x and such that $\langle [A^{<x}]_n, [A^{>x}]_n\rangle = \langle [A^{<y}]_n, [A^{>y}]_n\rangle$.*

Then A is $(n+1)$-equivalent to $B = A - (a, b]$.

We have another 'cutting lemma', relevant just for the case of limit ordinals, which is our main new tool over the finite case. Before we can prove this, an auxiliary result is required, which actually applies to all coloured linear orders, not just ordinals.

Lemma 3.2. *Let A be an m-coloured linear order and let $a_1 < a_2$ and $b_1 < b_2$ be elements of A such that a_1 and b_1 have the same n-characters, and so do a_2 and b_2, where $n \ge 1$, and such that the families of n-characters of members of (a_1, a_2) and (b_1, b_2) are equal to the same set C_n, and there are at least $2^n - 1$ blocks of occurrences of members of C_n in each of (a_1, a_2) and (b_1, b_2); where this means that (a_1, a_2) and (b_1, b_2) may each be written as the disjoint union of this number of convex subsets on each of which all members of C_n are represented. Then $(a_1, a_2) \equiv_n (b_1, b_2)$.*

Proof. We use induction. For the basis case, $n = 1$, so there is at least one block of occurrences of C_1 (which in this case is given anyhow by definition of C_1), and the information given by the character is just the colour. Player II can therefore win in one move by playing a point of the same colour as player I did.

Now assume the result for n, and we indicate how player II can play to win the $(n+1)$-move game between (a_1, a_2) and (b_1, b_2), assuming that there are at least $2^{n+1} - 1$ blocks of occurrences of members of C_{n+1} in each of (a_1, a_2) and (b_1, b_2). Without loss of generality player I starts by playing $x_1 \in (a_1, a_2)$.

First suppose that (a_1, x_1) and (x_1, a_2) each have at least $2^n - 1$ blocks of occurrences of members of C_{n+1}. Since $2^{n+1} - 1 = (2^n - 1) + 1 + (2^n - 1)$, player II can play a point y_1 (of the 'middle' block) having the same $(n+1)$-character as x_1. Now from the fact that (a_1, x_1) and (x_1, a_2) exhibit precisely the same $(n+1)$-characters it follows that they also exhibit the same n-characters, so the induction hypothesis assures us that $(a_1, x_1) \equiv_n (b_1, y_1)$ and $(x_1, a_2) \equiv_n (y_1, b_2)$, and so II can win.

Next suppose that (a_1, x_1) does not have $2^n - 1$ blocks of occurrences of members of C_{n+1} (and a similar argument applies if (x_1, a_2) does not have $2^n - 1$ blocks). Since a_1 and b_1 have the same $(n+1)$-character, $(a_1, \infty) \equiv_{n+1} (b_1, \infty)$, so there is $y \in (b_1, \infty)$ of the same colour as x_1 such that $(a_1, x_1) \equiv_n (b_1, y)$ and $(x_1, \infty) \equiv_n (y, \infty)$. Also $(-\infty, x_1) = (-\infty, a_1) \cup \{a_1\} \cup (a_1, x_1) \equiv_n (-\infty, b_1) \cup \{b_1\} \cup (b_1, y) = (-\infty, y)$, which shows that x_1 and y have the same n-character.

If (b_1, y) does not contain $2^n - 1$ blocks of occurrences of members of C_{n+1}, then $y < b_2$, and furthermore, each of (x_1, a_2) and (y, b_2) contains at least $2^n - 1$ blocks of occurrences of members of C_{n+1}. Since x_1 and y have the same n-character, we may apply the induction hypothesis to deduce that $(x_1, a_2) \equiv_n (y, b_2)$, and so II wins by playing $y_1 = y$ on his first move.

Otherwise (b_1, y) contains at least $2^n - 1$ blocks of occurrences of members of C_{n+1}. Player II now plays a point y_1 in the middle block of (b_1, b_2) having the same n-character as y (which lies in C_n since the n-characters of x_1 and y are equal). By the induction hypothesis, $(b_1, y) \equiv_n (b_1, y_1)$, and it follows that $(a_1, x_1) \equiv_n (b_1, y_1)$. Also both (x_1, a_2) and (y_1, b_2) contain at least $2^n - 1$ blocks of occurrences of members of C_{n+1}, so by induction hypothesis, they are n-equivalent. By applying the same argument as in the previous paragraph, x_1 and y_1 have the same n-character, so player II wins by playing this y_1. ∎

We can now present our main new 'cutting' lemma.

Lemma 3.3. *Let A be an m-coloured ordinal, Λ an ordinal, and for each $\lambda \in \Lambda$ let a_λ and b_λ be elements of A which are limit ordinals such that $\lambda < \mu \Rightarrow a_\lambda < b_\lambda < a_\mu$ and for each limit ordinal $\lambda \in \Lambda$, $\sup_{\mu<\lambda} b_\mu < a_\lambda$. Suppose further that the sets of n-characters which occur cofinally in $(-\infty, a_\lambda)$ and $(-\infty, b_\lambda)$ are equal to the same set C_λ, and the n-characters of all points of $[a_\lambda, b_\lambda)$ also lie in C_λ. Then A is $(n+1)$-equivalent to $B = A - \bigcup_{\lambda\in\Lambda}[a_\lambda, b_\lambda)$.*

Proof. First we choose $c_\lambda < a_\lambda$ so that all n-characters arising in $[c_\lambda, a_\lambda)$ (and hence also in $[c_\lambda, b_\lambda)$) lie in C_λ. This is possible since there are only finitely many characters in all, so there is some point $< a_\lambda$ beyond which any n-characters which do *not* occur cofinally in $(-\infty, a_\lambda)$ no longer arise. Furthermore, the hypothesis allows us to suppose that $\sup_{\mu<\lambda} b_\mu < c_\lambda$.

We describe a winning strategy for player II in the $(n+1)$-move game on A and B. We write x_i and y_i for the ith moves played in A, B respectively. The map taking x_i to y_i will be order-preserving (and all x_i will be distinct). Furthermore, if x_i, y_i have been chosen for $i \le k$, and I is an open interval determined by adjacent $x_i, x_{i'}$ or between x_i and $\pm\infty$, and J is the corresponding interval determined by the y_i, then $I \equiv_{n+1-k} J$. Also, x_i, y_i will have the same colour.

Player II can clearly play on his first move so that x_1 and y_1 have the same n-characters, and either $x_1 = y_1$, or $x_1 \in [a_\lambda, b_\lambda)$ and $y_1 \in (c_\lambda, a_\lambda)$. The fact that this is possible follows from the choice of c_λ, and the cofinality hypotheses.

Now suppose that x_i, y_i have been chosen for $i \le k$, and we have to say how player II can respond to any possible move by player I on his $(k+1)$th move.

Let I or J be the interval that I decides to play in (if he plays in A or B respectively). By assumption, $I \equiv_{n+1-k} J$, and we consider the response to I's play made by player II using a strategy thereby given. Let x_{k+1} and y be the moves thus played. If $y \in B$ we just let $y_{k+1} = y$, and all hypotheses carry through to the next step. If however $y \notin B$ (in which case player I must have played x_{k+1}) then for some λ, $y \in [a_\lambda, b_\lambda)$. By cofinality of the occurrences of points of n-character lying in C_λ in (a_λ, b_λ) we may find a point y_{k+1} of $(\max(y_i, c_\lambda), a_\lambda)$ having the same n-character as y, and with sufficiently many blocks of occurrences of C_λ in $(\max(y_i, c_\lambda), y_{k+1})$. II plays this y_{k+1} on his $(k+1)$th move. Since x_{k+1} and y have the same $(n-k)$-character and y and y_{k+1} have the same n-character, it follows that x_{k+1} and y_{k+1} have the same $(n-k)$-character. The fact that $(x_i, x_{k+1}) \equiv_{n-k} (y_i, y_{k+1})$ and $(x_{k+1}, x_{i'}) \equiv_{n-k} (y_{k+1}, y_{i'})$ follows from Lemma 3.2, so the induction goes through. ∎

The main theorem is now as follows:

Theorem 3.4. *For any positive integers m and n, there is a finite k such that for any m-coloured ordinal there is an n-equivalent m-coloured ordinal less than $\omega^k \cdot 2k^2$.*

Proof. The case $n = 1$ is easy and completely described in [4]. In fact, two coloured linear orders are 1-equivalent if and only if they exhibit precisely the same sets of colours, so we get (finite) optimal representatives of size at most m.

Moving on to $n > 1$, let $(A, <, F)$ be an m-coloured ordinal of minimal order-type in its $\equiv_n$-class. We start by considering the occurrences of the $(n-1)$-characters appearing in A. There are just finitely many, which from now on we refer to just as 'characters' and so we may find the first occurrence of each, and let these be $x_0 < x_1 < x_2 < \ldots < x_{k-1}$ (where clearly x_0 and x_1 are the first two members of A). For ease we also let $x_k = +\infty$, so that we can refer to the intervals $I_i = [x_i, x_{i+1})$ for all $i < k$.

Now by choice of the x_i as the *first* occurrences of the characters, any character arising in (x_i, x_{i+1}) already occurs in $(-\infty, x_i]$. Hence if any character arises more than once in I_i, then we may use Lemma 3.1 to cut out the section in between. Unlike in the finite case, this may however not reduce the order-type, but it does enable us to make some deductions about the form that A has, or may be assumed to have. Let us assume then that all characters of I_i appear with minimal order-type. For a particular character, write the order-type of its occurrences in I_i in Cantor normal form as $\omega^{\alpha_0} \cdot a_0 + \omega^{\alpha_1} \cdot a_1 + \ldots + \omega^{\alpha_l} \cdot a_l$ where $\alpha_0 > \alpha_1 > \ldots > \alpha_l$ and $a_i \in \omega$. If $l > 0$ then we may cut a section from $\omega^{\alpha_0} \cdot a_0$ to $\omega^{\alpha_l} \cdot a_l$ and achieve a strictly smaller order-type, contrary to assumption. Hence $l = 0$. A similar argument applies if $a_0 > 1$. We therefore deduce that the character appears with order-type of the form ω^α for some ordinal α (which is 1 if $\alpha = 0$).

We note that it also follows that I_i is expressible as a finite union of intervals of the form $J_j = [y_j, y_{j+1})$ where no character appears in more than one J_j, and each character of J_j appears cofinally (the case $|J_j| = 1$ is allowed). To achieve this, the points y_{j+1} are taken to be the suprema of the occurrences of characters. Given this, to see that no character appears in more than one J_j, observe that no character which appears in $[y_{j+1}, x_{i+1})$ can also appear in (x_i, y_{j+1}). For if it did, we could apply Lemma 3.1 and reduce the order-type of the occurrences of the characters having supremum y_{j+1}, contrary to the assumption that the order-types of occurrences of all characters have been minimized.

We emphasize that the same character can (and will) occur in more than one of the intervals I_i, but for *fixed* i, no character will occur in more than one J_j.

To conclude the proof, we show by induction on $r \geq 1$ that J_j has a subset B_r such that $A \equiv_n A - B_r$ and for any non-empty set X of r characters, all convex subsets of $J_j - B_r$ exhibiting only members of X have order-type $< \omega^r \cdot 2$.

For the basis case, $r = 1$, and let c be a single character. Define $\sim$ on J_j by $x \sim y$ if $x = y$, or if all points of $[x, y]$ (or $[y, x]$ if $y < x$) have character c. Then the $\sim$-classes are convex subsets of J_j. Let $[\beta, \gamma)$ be a $\sim$-class, and λ_1 its least limit ordinal, if any, and $\gamma = \lambda_2 + s$ for finite s, where λ_2 is a limit ordinal (or 0). We may apply Lemma 3.3 to cut out $[\lambda_1, \lambda_2)$ from all $\sim$-classes containing a limit ordinal, giving a subset in which all $\sim$-classes have order-type $< \omega \cdot 2$. Note that the requirement that the supremum of the right hand endpoints of the cut out intervals is strictly less than the next one is automatically fulfilled, since each λ_1 is immediately preceded by a non-empty block of points all having character c. Now repeat this for each of the remaining characters and let B_1 be the union of all the sets cut out.

For the induction step, assume that we have found B_r and that $1 \leq r < k$. Let X be a set of characters of size $r + 1$, and define $\sim$ on $J_j - B_r$ by letting $x \sim y$ if $x = y$, or if all points of $[x, y]$ (or $[y, x]$ if $y < x$) have character in X. By appropriately cutting segments from each $\sim$-class, they will have the form $[\beta, \gamma)$ where all members of X are cofinal in at most one limit ordinal in $[\beta, \gamma]$. This means that for all other limit ordinals λ in this interval, the set of characters which occur cofinally in $[\beta, \lambda)$ is a proper subset of X. We show that the order-type of $[\beta, \gamma)$ is less than $\omega^{r+1} \cdot 2$. If not, then there is a limit ordinal λ in $[\beta, \gamma]$ such that $[\delta, \lambda) \cong \omega^{r+1}$ for some $\delta \in [\beta, \lambda)$ and such that the set Y of characters which occur in $[\delta, \lambda)$ is a proper subset of X, but this contradicts the induction hypothesis. Now repeat this argument finitely many times for all sets of characters of size $r + 1$, and this gives the induction step by taking for B_{r+1} the union of all the sets removed at these finitely many steps.

Finally we look at the case where $r = k$ which has now been established. We have a set of order-type less than $\omega^k \cdot 2$ which is n-equivalent to J_j. This gives a bound $\omega^k \cdot 2k^2$ for the order-type of A, where k is the number of all $(n-1)$-characters. This is an explicit bound, but since the number of $(n-1)$-characters grows very fast, we believe that it is much greater than the optimum. ■

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